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NUMBER THEORY

Apéry, Roger. Un théorème d'arithmétique. C. R. Acad. Sci. Paris 219, 404–405 (1944). [MF 15273]

For integers $a_1, \cdots, a_n$ define $P=\prod a_i$ and $S=\sum a_i^2$. It is proved that for any $n\geqq 3$ the integers a_i can be chosen in infinitely many ways so that they are relatively prime in pairs and the quotient S/P is an integer not greater than n.

I. Niven (West Lafayette, Ind.).

Vijayaraghavan, T., and Chowla, S. The complete factorization (mod p) of the cyclotomic polynomial of order p^2-1. Proc. Nat. Acad. Sci. India. Sect. A. 14, 101–105 (1944). [MF 14292]

The purpose of this paper is to show that for certain primes p the irreducible cyclotomic polynomial whose roots are the (p^2-1)th roots of unity is decomposable modulo p into certain quadratic factors. More precisely, the theorem proved may be stated as follows. Let p be a prime such that $(p+1)/2$ is either a prime or a power of 2. Then, if $F(x)$ is the polynomial mentioned above,

$$F(x)\equiv\prod_{a,b}(x^2+ax+b) \pmod p,$$

where b runs over the primitive roots of p and, for each b, a runs over all incongruent values (mod p) for which a^2-4b is a quadratic nonresidue of p.

D. H. Lehmer.

Pall, Gordon. Note on factorization in a quadratic field. Bull. Amer. Math. Soc. 51, 771–775 (1945). [MF 13618]

The author considers the problem of factorization of integers in quadratic fields $R(\sqrt{\Delta})$ with $\Delta\equiv\epsilon\equiv 0$ or 1 (mod 4). He employs the integral basis 1, $\rho=(\Delta^{\frac{1}{2}}-\epsilon)/2$ and shows that generally a primitive integer $x=x_0+x_1\rho$ (that is, one with x_0, x_1 coprime) has at most one factorization into primes. Next he studies conditions for the existence of factors of given norm and remarks that, in view of the above mentioned uniqueness of factorization as well as the analogous results in the arithmetic of quaternions, one may regard the introduction of ideals as restoring "existence" of factorization into primes.

A. E. Ross (Saint Louis, Mo.).

Uhler, H. S. Note on the Mersenne numbers M_{127} and M_{257}. Bull. Amer. Math. Soc. 52, 178 (1946). [MF 15459]

This note gives the information on $2^{127}-1$ already announced [Mathematical Tables and Other Aids to Computation 1, 404 (1945); see also the review of a note by C. B. Barker, these Rev. 6, 255]. There is also announced a complete agreement between the author's results proving that $2^{257}-1$ is composite and those of C. B. Barker [see Uhler, Proc. Nat. Acad. Sci. U. S. A. 30, 314–316 (1944); these Rev. 6, 57].

D. H. Lehmer.

Ramanathan, K. G. Congruence properties of $\sigma_a(n)$. Math. Student 13, 30 (1945). [MF 14492]

The author states and illustrates the divisibility properties of the function $\sigma_a(n)$ mentioned in the following review.

D. H. Lehmer (Aberdeen Proving Ground, Md.).

Gupta, Hansraj. Congruence properties of $\sigma(n)$. Math. Student 13, 25–29 (1945). [MF 14491]

Let $\sigma_a(n)$ denote, as usual, the sum of the ath powers of the divisors of n. Ramanathan has noted [Math. Student 11, 33–35 (1943); these Rev. 6, 37] that, for every $m>0$, $\sigma_1(km-1)$ is divisible by k whenever $k>2$ and is a divisor of 24. The present paper shows that this property holds for no other value of k. For the more general function $\sigma_a(n)$, the author proves the following result noted by Ramanathan [see the preceding review]. In order that $\sigma_a(km+r)$ be divisible by k for every $m>0$ it is first necessary that r^a+1 be divisible by k; for such r it is then both necessary and sufficient that $t^{2a}-1$ be divisible by k for every t prime to and less than k.

D. H. Lehmer.

Oblath, Richard. On impossible Diophantine equations of the form $x^n+1=y^n$. Revista Mat. Hisp.-Amer. (4) 1, 122–140 (1941). (Spanish) [MF 12994]

The principal theorems proved are the following. (I) The Diophantine equation $x^n-1=y^n$ is impossible in nonzero integers if n is a prime $(n>3)$ satisfying either of the conditions $2^{n-1}\not\equiv 1 \pmod{n^2}$ or $3^{n-1}\not\equiv 1 \pmod{n^2}$ (hence if $3<n<16000$). (II) Hence the equation is impossible (save for trivial cases) if n is a prime factor of numbers of the form $2^a\pm 1$, $2^a3^\beta\pm 1$, $2^a\pm 3^\beta$ or $3^\beta\pm 1$. (III) The equation is impossible if either (i) $y=2^ap$ (p prime), (ii) $y=2^a3^\beta5^\gamma$ or (iii) y is a product of primes less than 32000. (IV) The equation $x^n+1=y^n$ is impossible if y consists only of primes of the form 2^a+1 or $2^a3^\beta+1$. Gérono [Nouv. Ann. Math. (2) 9, 469–471 (1870); 10, 204–206 (1871)] had considered the last equation with x or y prime. Use is made of the work of Nagell on $x^3-1=y^n$ [latest reference, Norsk. Vidensk. Selsk. Forh. 7, 136–139 (1934)] and Lubelski on $x^n+y^n=cz^n$ [Prace Mat. Fiz. 42, 11–34 (1935), in particular, p. 29].

G. Pall (Montreal, Que.).

van IJzeren, J. Elementary properties of the partitions of the natural numbers. Mathematica, Zutphen. B. 12, 115–118 (1944). (Dutch) [MF 15720]

Erdös, Paul. On some asymptotic formulas in the theory of partitions. Bull. Amer. Math. Soc. 52, 185–188 (1946). [MF 15461]

Let $p_k(n)$ denote the number of partitions of n into exactly k summands and let k_0 be that value of k for which $p_k(n)$ is greatest. The formula $k_0=c^{-1}n^{\frac{1}{2}}\log n+an^{\frac{1}{2}}+o(n^{\frac{1}{2}})$, $c=\pi(2/3)^{\frac{1}{2}}$, $c/2=e^{-a\sqrt{3}/2}$, is proved. This result includes the conjecture of Auluck, Chowla and Gupta [J. Indian Math. Soc. (N.S.) 6, 105–112 (1942); these Rev. 4, 211] that $k_0\sim c^{-1}n^{\frac{1}{2}}\log n$.

H. S. Zuckerman (Seattle, Wash.).

Jogin, I. I. Zur Theorie der Diophantischen Approximationen. Uchenye Zapiski Moskov. Gos. Univ. Matematika 73, 37–40 (1944). (Russian. German summary) [MF 15191]

[In the original, the author's name was transliterated Schogin. The Russian spelling is Žogin.] The following the-

orem is proved. Let θ be irrational and let $F(t)>0$ be an increasing function for $t\geqq 1$ such that $F(2t)/F(t)$ is bounded as $t\to\infty$. There exists a constant $\Gamma>0$, such that $1\leqq x<\Gamma F(t)$, $|x\theta-y-\alpha|<1/t$ is soluble in integral x, y for all $t\geqq 2$ and all real α if and only if there exists a constant $\gamma>0$ such that $|q\theta-p|\geqq\gamma/F(q)$ for all integral $p, q\geqq 1$. [The reviewer remarks that this is a special case of an analogous theorem in n dimensions, which can be proved, for example, by means of his "Übertragungssatz"; see Nederl. Akad. Wetensch., Proc. 41, 634–637 (1938); Časopis Pěst. Mat. Fys. 68, 93–102 (1939); these Rev. 1, 202]. *K. Mahler.*

Jogin, I. I. Über eine Frage der Theorie der Diophantischen Approximationen. Uchenye Zapiski Moskov. Gos. Univ. Matematika 73, 41–44 (1944). (Russian. German summary) [MF 15192]

A proof that the inequality $|x\theta-y-\alpha|<5^{-1}|x|$ has an infinity of integral solutions if θ is irrational and α an arbitrary real number. [The author adds that 5^{-1} is the best possible constant; this is not correct unless x is restricted to positive integers. Compare Koksma, Diophantische Approximationen, Ergebnisse der Math. 4, no. 4, Springer, Berlin, 1936, chapter 6, §§ 2–3.] *K. Mahler* (Manchester).

Pipping, Nils. Approximation mehrerer reellen Zahlen durch rationale Zahlen mit gemeinsamem Nenner. Acta Acad. Aboensis 13, no. 9, 12 pp. (1942). [MF 15099]

Dirichlet proved that, if $\xi_1, \cdots, \xi_n$ denote $n\geqq 1$ arbitrary real numbers and m denotes any positive integer, n fractions $p_1/q, \cdots, p_n/q$ $(q\geqq 1)$ exist, such that $|\xi_\nu-p_\nu/q|\leqq q^{-m}$ $(\nu=1, \cdots, n)$, $q\leqq m^n$ [cf. the reviewer's Diophantische Approximationen, Ergebnisse der Math. 4, no. 4, Springer, Berlin, 1936, chapters 1, 4]. By a refinement of Dirichlet's argument ("Schubfachprinzip") the author proves that, if $m\geqq 2$, the inequality $q\leqq m^n$ can be replaced by $q\leqq m^n-2^n+1$. By a geometrical interpretation of the method he proves a further improvement in the case $n=2$, $m=3$, namely $q\leqq 5$. Finally, he considers the approximation $\sum_{\nu=1}^{n}|q\xi_\nu-p_\nu|\leqq 1$. From the above theorem it follows that this approximation always can be realized with integers $p_1, \cdots, p_n, q$ such that $1\leqq q\leqq n^n-2^n+1$. Improvements are given in special cases. *J. F. Koksma* (Amsterdam).

Ballieu, Robert. Sur le développement des irrationnelles quadratiques en fractions continues régulières. Mathesis 54, 304–314 (1942). [MF 15529]

Let $a_0x^2+b_0x+c_0=0$ have integral coefficients, a positive discriminant not a perfect square, and a root x_0. The complete quotients x_i and the incomplete quotients n_i of the development of x_0 into a continued fraction are given by $x_i=n_i+1/x_{i+1}$, n_i integral, $x_{i+1}>1$. The classical theorems of Lagrange, Legendre, Galois and Serret concerning the properties of these periodic continued fractions are proved by the writer. His approach differs from the classical one in that he studies, for the most part, not the continued fractions themselves, but the equations $a_ix^2+b_ix+c_i=0$ having roots x_i. Thus the discussion centers around the transformations which carry the ith such equation into the $(i+1)$th. The method seems to be briefer than that usually given. *I. Niven* (West Lafayette, Ind.).

Huzurbazar, V. S. On a property of rational numbers. J. Univ. Bombay (N.S.) 14, part 3, 1–3 (1945). [MF 15673]

The property is that all rational numbers sufficiently close to a given real positive number (except perhaps for the number itself) have arbitrarily large numerators and denominators. *R. P. Boas, Jr.* (Providence, R. I.).

Roussel, André. Sur un procédé de traduction analytique de propriétés arithmétiques. C. R. Acad. Sci. Paris 219, 568–569 (1944). [MF 15288]

The writer continues his work [same C. R. 217, 39–41, 496–497 (1943); these Rev. 5, 254; 6, 118] on equivalences between analytic and number-theoretic statements. He gives a necessary and sufficient condition that p is a prime and a statement equivalent to Fermat's last theorem, both involving integrals. *I. Niven* (West Lafayette, Ind.).

Vinogradow, I. Analytical theory of numbers. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 159–168 (1945). (Russian and English) [MF 13326]

This article was written on the occasion of the 220th anniversary of the Russian Academy of Sciences. It recounts the outstanding role played by Russian mathematicians in the development of the analytic theory of numbers, in particular, Euler, Chebyshev and Voronoi. Voronoi's work on the asymptotic distribution law for the number of divisors of n gave an impulse to Vinogradow's own first researches. A survey is given of the evolution of Vinogradow's researches, especially in connection with the Waring problem. Recent results by his pupils are mentioned. Thus Tchoudakov has improved the remainder term in the asymptotic formula for $\pi(x)$ to $\pi\exp\{(\log x)^\delta\}$, where $\delta=.6-\epsilon$; Linnik has shown that $\pi(x; a, b)>0$ for all x of the order a^c, where c is a constant. *G. Pall.*

Ross, Arnold E. On a problem of Ramanujan. Amer. J. Math. 68, 29–46 (1946). [MF 15487]

The author considers the problem of determining the conditions under which general integral positive quaternary quadratic forms represent all except a finite number of integers. If the leading principal minors of the integral symmetric matrix $A=(a_{ij})$ of a quaternary quadratic form $f=\sum a_{ij}x_ix_j$ with determinant $|a_{ij}|=|A|$ are denoted by A_1 $(=a_{11})$, o_1A_2 $(=a_{11}a_{22}-a_{12}^2)$ and $o_1^2o_2A_3$ (where the o_κ are the Minkowski-Smith invariants) he proves that every properly primitive classic quaternary form is equivalent to a canonical form f, where A_μ or $\frac{1}{2}A_\mu$ is an odd prime not dividing $|a_{ij}|A_\lambda A_l$ for $(\mu, k, l)=(1, 2, 3), (2, 1, 3), (3, 1, 2)$. Starting from a canonical form f, the form $A_\lambda A_\mu A_\nu f$ is replaced by an equivalent form G without the cross products. If f represents an integer m, then G represents $A_\lambda A_\mu A_\nu m$. If $(o_1o_2A_1, A_3)=(o_1, A_3)=1$, the converse is also true. For properly primitive canonical forms f of odd determinants the latter conditions are satisfied. The problem is thus reduced to the case of forms without the cross products. For this special case the problem has been solved (save for a finite number of exceptions) by the reviewer [Acta Math. 49, 407–464 (1926)]. The author shows that the reviewer's conditions assuring representation of all large integers are restrictions upon the generic characters of f. Some of these conditions are also necessary. However, the failure of the remaining conditions merely implies that a form represents all large integers only if it represents all positive integers or all even positive integers. The author shows that the determinants of such forms do not exceed a fixed number. For universal quadratic forms the exact upper bound is 112. Thus there is only a finite number of classes of forms of odd determinants representing all large integers and not satisfying the above generic character conditions. Indeed, there are only three such classes representing all positive integers.

These three classes are explicitly determined. Two of them are the only classes in their respective genera.

H. D. Kloosterman (Leiden).

Pall, Gordon. The completion of a problem of Kloosterman. Amer. J. Math. **68**, 47–58 (1946). [MF 15488]

The problem of determining all forms $f=(a, b, c, d) = ax^2+by^2+cz^2+dt^2$ (where a, b, c, d are positive integers) which represent all but a finite number of positive integers was solved by the reviewer, except that he was unable to decide whether the four forms (1, 2, 11, 38), (1, 2, 17, 34), (1, 2, 19, 22), (1, 2, 19, 38) represent all large integers [Acta Math. **49**, 407–464 (1926)]. The author proves that these forms do in fact represent all large integers. He gives an elegant formulation of the reviewer's conditions [loc. cit., pp. 453–454] assuring representation of all large integers. If p is any prime, a form f is said to be p-adically universal when $f\equiv n \pmod{p^r}$ is solvable in integers x, y, z, t for every n and $r\geqq 0$. The form f is said to represent zero p-adically if the congruence $f\equiv 0 \pmod{p^r}$ is solvable for every positive integer r in integers x, y, z, t not all divisible by p. The author gives criteria for p-adic universality and criteria for p-adic representation of zero (the latter being essentially those of Hasse [J. Reine Angew. Math. **152**, 129–148 (1923)]). The following result is then obtained. If (a) f is p-adically universal for every p and (b) f represents zero p-adically for every p, then f represents all sufficiently large integers. This follows from the following theorem. Let n denote any integer such that $f\equiv n \pmod{k}$ is solvable for every modulus k. If for each prime p such that f fails to represent zero p-adically an upper bound is imposed to the power of p in n, then f represents every sufficiently large positive n thus restricted. This theorem follows from the reviewer's asymptotic formula for the number $f(n)$ of representations of n by f. It is clear that the condition (a) is also necessary for a form f to represent all large integers. There are, however, exactly 199 forms which fail to represent zero p-adically for some p and yet represent all large integers.

H. D. Kloosterman (Leiden).

Ross, Arnold E., and Pall, Gordon. An extension of a problem of Kloosterman. Amer. J. Math. **68**, 59–65 (1946). [MF 15489]

The following theorem is proved. Let f be any positive m-ary quadratic form, $m\geqq 4$. Let n be such that $f(x_1, \cdots, x_m) \equiv n \pmod{p^r}$ is solvable for every p and r. For each prime p (if any) such that f fails to represent zero p-adically [see the preceding review] impose an upper bound to the power of p which may divide n. Then f represents every such n sufficiently large. The proof makes use of an extension to a general positive m-ary form ($m\geqq 4$) of the asymptotic formula for the number of representations of n in f which the reviewer proved [Acta Math. **49**, 407–464 (1926)] for the special case of a quaternary form $ax^2+by^2+cz^2+dt^2$ without the cross products. This extension is, however, a corollary of a theorem which the reviewer proved [Abh. Math. Sem. Hamburgischen Univ. **5**, 337–352 (1927)]. The authors prove it by induction, starting from the special case without cross products. Conditions for p-adic universality and p-adic representation of zero for quaternary forms are given and the following theorem is proved. The number of classes of integral positive quaternary forms which represent all large integers and yet fail to represent zero p-adically for some prime p is finite. The authors give an upper bound for the determinants of such forms. This enables the determination of all positive quaternary quadratic forms which represent all large integers.

H. D. Kloosterman (Leiden).

de Vries, G. W. A theorem of Minkowski. Mathematica, Zutphen. B. **12**, 119–125 (1944). (Dutch) [MF 15721]

Exposition of results of Davenport [Quart. J. Math., Oxford Ser. **10**, 119–121 (1938)].

Brun, Viggo. Méthode élémentaire pour évaluer des fonctions énumératives. I. Skr. Norske Vid. Akad. Oslo. I. **1941**, no. 12, 14 pp. (1942). [MF 13359]

This article studies the function

$$D(x)=[x/1]+[x/2]+[x/3]+\cdots,$$

that is, the number of lattice points within the hyperbola $uv=x$. Use is made of the symbols

$$L(x^s\psi(s))=\lim_{s\to\infty}\left(\frac{2\psi(s)x^s}{1!}-\frac{3\psi(2s)x^{2s}}{2!}+\frac{4\psi(3s)x^{3s}}{3!}-\cdots\right),$$

$$UA(x)=\int_1^x A(x/u)du-\sum_{u=1}^{x}A(x/u)+\tfrac{1}{2}A(x),$$

and the expression $[x]=\phi(x)+\phi(x/2)+\phi(x/3)+\cdots$, where $\phi(x)=1$ if $x\geqq 1$, $\phi(x)=0$ if $0\leqq x<1$. The final result is

$$x^{-1}\int_0^x D(x)dx=\tfrac{1}{2}x\log x+(E-\tfrac{1}{2})x+\tfrac{1}{4}+\theta(\tfrac{1}{8}\log x+\tfrac{1}{12}+3/x),$$

or the equivalent formula for $\Delta(x)=D(x)-x\log x-(2E-1)x$,

$$x^{-1}\int_0^x \Delta(x)dx=\theta(\tfrac{1}{8}\log x+\tfrac{1}{3}+3/x),$$

valid for $x>1$. Here E is Euler's constant and $-1<\theta<1$. Cramér's formula $\frac{1}{4}+O(x^{-1})$ for the mean value of $\Delta(x)$ was only asymptotic.

G. Pall (Montreal, Que.).

Skolem, Th. Über Nebenkörper und Nebenringe. Skr. Norske Vid. Akad. Oslo. I. **1944**, no. 6, 70 pp. (1944). [MF 13895]

By a cofield (Nebenkörper), the author means a set $k'=k\alpha$, where k is an algebraic number field and α an algebraic number, $\alpha\neq 0$. The product of two cofields $k\alpha$ and $k\beta$ belonging to the same k is again such a cofield, $k\alpha k\beta=k\alpha\beta$. It follows easily that the cofields with a fixed k form a group in which k is the unit element. If $\mathfrak{o}$ is the ring of all integers of k, the set $\mathfrak{o}'$ of all integers of $k'=k\alpha$ is a finite $\mathfrak{o}$-module. An ideal $\mathfrak{m}'$ of $\mathfrak{o}'$ is defined as an $\mathfrak{o}$-submodule of $\mathfrak{o}'$. Every such $\mathfrak{m}'$ consists of those numbers of $\mathfrak{m}'$ which are divisible by a fixed algebraic integer μ. It will in general be impossible to choose μ as a number of $\mathfrak{o}'$. A cofield is termed a proper cofield when 1 is the greatest common divisor of the integers of the cofield. The proper cofields belonging to a fixed k form a group under multiplication. Consider a fixed subgroup $\mathfrak{G}$ of this group. If k_r, k_s, k_t are three cofields which are elements of $\mathfrak{G}$ and if $k_rk_s=k_t$ then an ideal $\mathfrak{b}_r$ of k_r and an ideal $\mathfrak{b}_t$ of k_t are said to be associated when $\mathfrak{b}_r\mathfrak{o}_s=\mathfrak{b}_t$, where $\mathfrak{o}_s$ denotes the set of integers of k_s. We have here an equivalence relation. Furthermore, every ideal of k_r is associated with an ideal of k. The known properties of the ideals of the field k can be used to obtain corresponding statements for the ideals of the cofields in $\mathfrak{G}$. For instance, a unique decomposition of ideals into prime ideals can be established; classes of ideals can be discussed. A cofield k_r of $\mathfrak{G}$ may be written in the form $k\mu^{-1}$, where μ is an algebraic integer. If $\mathfrak{o}_r$ is the set of all integers of k_r, then $\mathfrak{a}_r=\mu\mathfrak{o}_r$ is an ideal of k. The cofield k_r contains units if and only if

$\mathfrak{a}_r$ is a principal ideal. An ideal in the reciprocal cofield to k_r is a principal ideal if it belongs to the same class as $\mathfrak{a}_r$.

Next, the set theoretical sum K of all the k_r in $\mathfrak{G}$ is formed; let O denote the set of integers in K. A K-module is defined as a subset M of K such that, if α and β belong to M and also belong to the same k_r, then $\alpha-\beta$ also belongs to M. A multiplication of these K-modules can be defined. A K-module M is termed an O-ideal when $M \subseteq O$, $MO \subset M$. If we mean by the components of a K-module M the intersections of M with the different k_r, then the components of an O-ideal M are associated ideals. Now the results obtained for ideals in the cofields can be used to obtain analogous results for O-ideals. For instance, classes of O-ideals can be introduced. If the ideal $\mathfrak{a}_r$ of $\mathfrak{o}$ is defined as above, let C_r be the class of ideals of $\mathfrak{o}$ which contains the ideal $\mathfrak{a}_r$. These elements C_r generate a subgroup H in the group G of all classes of $\mathfrak{o}$. The group of O-classes then is isomorphic with G/H. On the other hand, the cofields k_r which contain units form a subgroup $\mathfrak{G}'$ of $\mathfrak{G}$, and $\mathfrak{G}/\mathfrak{G}' \cong H$. The given group $\mathfrak{G}$ of cofields k_r is said to be a natural group of cofields if $\mathfrak{G}'$ is the unit element k of $\mathfrak{G}$, and if $G=H$. For every k, such natural groups exist; there is a finite number of them; they can all be constructed explicitly. The class number in O is 1; the laws for the divisibility of integers in O are analogous to the laws governing the divisibility of rational integers. The result is closely related to the well-known theorem that, to every algebraic number field k, there exists an algebraic extension field in which every ideal of k becomes a principal ideal. It may be remarked that the proofs of all the preceding theorems are not very complicated.

In the following chapter, the considerations are extended to the case where $\mathfrak{o}$ is replaced by a subring $\mathfrak{g}$. As an application, a proof is given for the formula which connects the class numbers of $\mathfrak{o}$ and of $\mathfrak{g}$. Later, discriminants, norms and decomposable forms are discussed. A method is developed which allows the results to be established in many cases without using the theory of ideals. The results are applied to the study of some Diophantine equations. Finally, the author gives some remarks concerning the generalization of his results to noncommutative cases. As an example, a certain skew field consisting of generalized quaternions is discussed in detail. *R. Brauer* (Toronto, Ont.).

ANALYSIS

Bang, Thøger. Une inégalité de Kolmogoroff et les fonctions presque-périodiques. Danske Vid. Selsk. Math.-Fys. Medd. **19**, no. 4, 28 pp. (1941). [MF 15391]

Let $f(x)$ be a real function defined and bounded, together with its derivatives of orders $1, 2, \cdots, n$, on the real axis, and let M_k denote the least upper bound of $|f^{(k)}(x)|$. Kolmogoroff has given the (best possible) inequality

$$(*) \qquad (M_k/M_0^{k})^{1/k} \leqq (M_n/M_0^{n})^{1/n}, \qquad 0<k<n,$$

where

$$t_k = (4/\pi)\{1+(-3)^{-k+1}+5^{-k+1}+(-7)^{-k+1}+\cdots\}$$

[C. R. Acad. Sci. Paris **207**, 764–765 (1938); Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 3–16 (1939); these Rev. **1**, 298]. The author gives a new proof of (*) by establishing it first for almost-periodic functions and then obtaining the general case by approximation. He also shows that (*) leads simply to the Bohr-Favard inequality for the integrals, and to the Bernstein inequality for the derivatives, of trigonometric sums. He then extends (*) to derivatives of fractional order in the following way. Let $f_\gamma(x)$, $f_\beta(x)$ be (uniformly) almost periodic functions with Fourier series $\sum a_\lambda(i\lambda)^{-\gamma}e^{i\lambda x}$, $\sum a_\lambda(i\lambda)^{-\beta}e^{i\lambda x}$, $\gamma^*<\beta^*$ (where the star denotes the real part). Then if $\gamma^*<\alpha^*<\beta^*$ the series $\sum a_\lambda(i\lambda)^{-\alpha}e^{i\lambda x}$ is the Fourier series of an almost periodic function $f_\alpha(x)$ and, if M_k denotes the least upper bound of $f_k(x)$, we have

$$M_\alpha \leqq M_\gamma^{(\beta^*-\alpha^*)/(\beta^*-\gamma^*)} M_\beta^{(\alpha^*-\gamma^*)/(\beta^*-\gamma^*)} K_{\beta-\gamma}^{\alpha-\gamma},$$

where $K_{\beta-\gamma}^{\alpha-\gamma}$ depends only on $\alpha-\gamma$ and $\beta-\gamma$. As consequences, the author deduces generalized Bohr-Favard and Bernstein inequalities for integrals and derivatives of fractional order, extending and improving results, respectively, of B. v. Sz. Nagy [Ber. Verh. Sächs. Akad. Wiss. Math.-Phys. Kl. **91**, 3–24 (1939)] and Civin [Duke Math. J. **8**, 656–665 (1941); these Rev. **3**, 108]. *R. P. Boas, Jr.*

Hukuhara, Masuo. Sur la fonction $S(x)$ de M. E. Kamke. Jap. J. Math. **17**, 289–298 (1941). [MF 14959]

E. Kamke [S.-B. Heidelberger Akad. Wiss. **20**, no. 17 (1930)] has considered the class K of functions $S(x_1, \cdots, x_n)$ defined by the properties that the functions S are continuous throughout Euclidean n-space E_n and vanish at the origin and, for every set of n differentiable functions $x_j(t)$, the derivatives $D^{\pm}S$ exist and satisfy

$$(1) \qquad D^{\pm}S[x_1(t), \cdots, x_n(t)] \leqq S[x_1'(t), \cdots, x_n'(t)].$$

The author shows that a function S continuous throughout E_n and vanishing at the origin is a member of K provided that S satisfies (1) with $D^{\pm}S$ replaced by $\bar{D}^{\pm}S$; it follows that the derivatives $D^{\pm}S$ necessarily exist. He shows, furthermore, that the above class of functions coincides with the class of functions continuous in E_n, positively homogeneous of degree one and convex. *E. F. Beckenbach.*

Woods, Cecil L. A restricted class of convex functions. Bull. Amer. Math. Soc. **52**, 117–128 (1946). [MF 15451]

Let $f(x)$ be a positive continuous function for x in an open interval J and let $J(x, h)$ denote a subinterval of J with midpoint x and length $2h$. Let $I(f, x, h, \alpha)$ and $A(f, x, h, \alpha)$, $-\infty<\alpha<+\infty$, denote the means of order α of $f(x)$, at the end-points of $J(x, h)$ and over the interval $J(x, h)$, respectively. Radó [Trans. Amer. Math. Soc. **37**, 266–285 (1935)] has determined continuous functions $\psi_2(\alpha)$, $\psi_3(\alpha)$ such that for all functions $f(x)$ convex in J we have, for all $J(x, h)$ in J,

$$(1) \qquad I(f, x, h, \alpha) \leqq A(f, x, h, \beta)$$

if and only if $\beta \geqq \max(\psi_2, \psi_3)$, while for all functions $f(x)$ concave in J we have (1) with the inequality reversed if and only if $\beta \leqq \min(\psi_2, \psi_3)$; on the other hand, (1) implies that $f(x)$ is convex if and only if $\beta \leqq \psi_2$, while (1) with the inequality reversed implies that $f(x)$ is concave if and only if $\beta \geqq \psi_2$. We have $\min(\psi_2, \psi_3) = \max(\psi_2, \psi_3)$ only at the intersection points $(\alpha, \beta) = (-2, 0), (-\frac{1}{2}, \frac{1}{2}), (1, 1)$: for these three pairs (1) holds with the sign of equality if and only if $f(x)$ is linear.

The author, in investigating pairs (α, β) for which $\min(\psi_2, \psi_3) < \beta < \max(\psi_2, \psi_3)$, is led to a consideration of a restricted class K_M of positive continuous functions in J defined by $\frac{1}{2}\log(B/b) \leqq M$, where B is the least upper bound and b the greatest lower bound of $f(x)$. For each positive M there is a function $\beta_M(\alpha)$ such that, for all functions of K_M convex in J, (1) holds if and only if $\beta \geqq \max(\psi_2, \beta_M)$,

while for all functions of K_M concave in J, the inequality is reversed if and only if $\beta \leqq \min(\psi_2, \beta_M)$. The functions $\beta_M(\alpha)$ satisfy $\min(\psi_2, \psi_3) \leqq \beta_M \leqq \max(\psi_2, \psi_3)$ for all α; $\beta_M(\alpha) \to \psi_2(\alpha)$ as $M \to 0$; $\beta_M(\alpha) \to \psi_3(\alpha)$ as $M \to \infty$. *E. F. Beckenbach.*

de Groot, J. Ueber die Fortsetzung differenzierbarer Funktionen. Mathematica, Zutphen. B. **12**, 15–24 (1943). [MF 15725]

The reviewer has shown [Trans. Amer. Math. Soc. **36**, 369–387 (1934)] that any function which is continuously differentiable (in a suitable sense) to the nth order on a closed subset of the real axis is extendable to have this property on the whole axis. K. Seebach [Math. Ann. **116**, 701–718 (1939)] proved a similar theorem, the function and its extension having a first derivative with at most a countable number of discontinuities. The present paper proves the theorem, the function and its extension having a first derivative with any number of discontinuities, and with infinite values of the derivative permitted. *H. Whitney.*

Yamanaka, Yukio. On a continuous tangent field. Jap. J. Math. **18**, 745–757 (1943). [MF 14982]

In a region of 2-dimensional Euclidean space a "continuous tangent field" is given, that is, an unoriented tangent direction at each point, the direction varying continuously with position. It is proved, but not too rigorously, that we can assign orientations so as to have a continuous vector field. The existence of curves (not uniquely determined) having the given tangent direction at each point is shown. The last part of the paper is devoted to consideration of whether a solution is a "simple curve" or "has no branch point" [but the terms do not mean what one would suppose]. Roughly, we have a regular point Q of a solution if other solutions through points near any point P of C in a neighborhood of Q do not meet C close to P; otherwise Q is a branch point. [The theorems are stated in language which seems to give properties of the solutions but actually give little information about an individual solution. Simple examples show that a point Q may be regular even though infinitely many solutions pass through Q.]

A. B. Brown (Flushing, N. Y.).

Theory of Sets, Theory of Functions of Real Variables

de Bruijn, N. G. Common representative systems of two divisions of an aggregate into classes. Nieuw Arch. Wiskunde (2) **22**, 48–52 (1943). (Dutch) [MF 15698]

This paper contains a proof of the following theorem, which constitutes a generalization of a theorem proved for finite sets by D. König [Math. Ann. **77**, 453–465 (1916)] and van der Waerden [Abh. Math. Sem. Hamburgischen Univ. **5**, 185–187 (1927)], for infinite sets by König and Valkó [Math. Ann. **95**, 135–138 (1925)]. If $\sum A$ and $\sum B$ represent two partitions of a set M, finite or infinite, which satisfy the conditions (I) for no natural number ν does a subset which is the join of $\nu-1$ sets A (B) contain ν sets B (A), (II) every set A (B) has elements in common with at most a finite number of sets B (A), then there exists a common representation of the two partitions, that is, a subset of M which contains exactly one element of each set A, and exactly one element from each set B. The proof follows, in the main, procedures used in the papers cited.

A. Dresden (Swarthmore, Pa.).

Cuesta, N. Continuous permutations with real numbers. Revista Mat. Hisp.-Amer. (4) **5**, 191–203 (1945). (Spanish) [MF 15207]

Examples of various dissimilar linearly ordered sets without gaps or jumps, having the cardinal number of the continuum, showing that 2^c are possible ($c = 2^{\aleph_0}$).

R. Arens (Princeton, N. J.).

Knaster, B. Sur une propriété caractéristique de l'ensemble des nombres réels. Rec. Math. [Mat. Sbornik] N.S. **16**(**58**), 281–290 (1945). (French. Russian summary) [MF 14583]

The author makes a careful study of certain aspects of the notion of separability. A series (ordered set) with neither jumps nor gaps is called continuous. A continuous series E is isomorphic with the real numbers in natural order if it has the property (D): there exists a countable subset dense in E. Souslin raised the problem of replacing (D) by (S): every class of intervals of E, no two of which have elements in common, is at most countable (a.m.c.). Knaster now enquires whether (D) can be replaced by (K): every more than countable (m.t.c.) class of intervals of E contains a m.t.c. subclass of which every two members have points in common. Clearly $(D) \to (K) \to (S)$. It is shown that $(K) \to (D)$ and so Souslin's problem, still unsolved, is equivalent to the proof that $(S) \to (K)$.

A general property of a class G of sets, related to several topological concepts, for example, that of weak separability [E. Szpilrajn, C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 525–527 (1941); these Rev. **3**, 57], can be formulated as (s): every subclass of G of which no two members have elements in common is a.m.c.; with this property we associate another (k): every m.t.c. subclass of G contains a m.t.c. subclass of which every two members have elements in common. Clearly $(k) \to (s)$. These properties remain significant when the relation of having elements in common is replaced by any symmetric relation. In this general case W. Sierpiński [Ann. Scuola Norm. Super. Pisa (2) **2**, 285–287 (1933)] has shown that the properties are not equivalent. The question of their equivalence in the case when G is the class of all open sets of a topological space remains open and appears important. As a contribution towards its understanding Knaster presents two further properties (s^*) and (k^*) which he proves equivalent, respectively, to (s) and (k) when G is any class of sets. These are: for every m.t.c. $X \subset G$ there exists a m.t.c. $X_1 \subset X$ such that (s^*): if $x \in X_1$ then $xy \neq 0$ for all $y \in X_1$ or such that (k^*): if $x \in X_1$ then $xy \neq 0$ for all $y \in X_1$ save perhaps for an a.m.c. set. *J. Todd* (London).

Arens, Richard. On the construction of linear homogeneous continua. Bol. Soc. Mat. Mexicana **2**, 33–36 (1945). [MF 14477]

A linear homogeneous continuum (LHC) is a simply ordered set in which every monotone sequence has a limit and which is isomorphic with every one of its closed subintervals (whose end-points are distinct). The author discusses properties of L^n, where L is an LHC, for instance, that L^n is itself an LHC, and concludes with some conjectures. *J. Todd* (London).

Kondô, Motokiti. Sur la structure des ensembles. Proc. Imp. Acad. Tokyo **18**, 57–64 (1942). [MF 14743]

An analytic operation $\Phi(\{E_n\})$ associates with a sequence of sets $E_n \subset R$ (where R is the set of irrational numbers) a set $\Phi(\{E_n\}) \subset R$ [cf. the author's papers in J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **7**, 1–34 (1938); **10**, 35–76

(1941); Proc. Imp. Acad. Tokyo **15**, 193–199 (1939); these Rev. **1**, 240]. Let $\Phi(X)$ denote the set of values of $\Phi(\{F_n\})$, where $\{F_n\}$ runs through all sequences of closed subsets of X. An analytic operation is said to be regular if for every representation $R=\sum_{n=1}^{\infty}R_n$, where each R_n is a G_δ and no two have points in common, the relation $E_n \varepsilon \Phi(R_n)$, $n=1, 2, \cdots$, implies $\sum_{n=1}^{\infty}E_n \varepsilon \Phi(R)$. Regular analytic operations can be defined which are such that $\Phi(R)$ is, for instance, the class of all α-elements in the Borel classification or the class of all analytic sets or the class of all sets whose complements are analytic sets.

It is shown that if Φ is a regular analytic operation then sets E having the two following properties exist and that any two of the sets are homeomorphic: (i) E is of the first category with respect to $\bar{E}$ (its closure); (ii) E is, on each portion of $\bar{E}$, a set universal for $\Phi(R)$ (that is, if $A \varepsilon \Phi(R)$ and P is a portion of $\bar{E}$ there is a closed set $F \subset P$ such that EF is homeomorphic with A). When Φ is such that $\Phi(R)$ is the class of α-elements these sets coincide with the canonical elements introduced by L. Keldych [Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 71–98 (1944); these Rev. **7**, 195], whose work was based on the notion of "transfinite equivalence." The proof of the present results is, however, based on a general lemma which is established by finite induction. New proofs are given for results already obtained by L. Keldych on the representation of Borel sets in terms of canonical elements and new results on the representation of analytic sets are stated. *J. Todd* (London).

Kondô, Motokiti. La structure des fonctions projectives. I. Proc. Imp. Acad. Tokyo **20**, 439–443 (1944). [MF 14910]

According to N. Lusin, a function is called projective if its geometric image is a projective set. The author gives the following equivalent but inductive definition of the projective functions $F(x)$ on a complete separable metric space R. The function $F(x)$ defined on R is called projective of the class A_1 or C_1, respectively, if there exists a Baire function $F(x, y)$, defined on the product space $R \times I$ (where I designates the set of all real numbers), satisfying the respective conditions

$$(1) \qquad \text{l.u.b.}_{-\infty<y<+\infty} F(x, y)=F(x), \qquad \text{g.l.b.}_{-\infty<y<+\infty} F(x, y)=F(x);$$

and $F(x)$ is called projective of the class B_1 if it is of the classes A_1 and C_1 simultaneously. Now assume the projective functions of the classes A_k, B_k, and C_k $(k=1, \cdots, n)$ to be defined already. Then the function $F(x)$ defined on R is called projective of the class A_{n+1} (or C_{n+1}) if there exists a function $F(x, y)$ of the class C_n (or A_n) defined on $R \times I$ and satisfying the appropriate one of conditions (1); $F(x)$ is called projective of the class B_{n+1} if it is of the classes A_{n+1} and C_{n+1} simultaneously.

The author then states relations between projective functions and projective sets. His main results refer to the composition of projective functions. He proves the following two theorems. (I) Let $G(t_1, \cdots, t_m)$ be a Baire function of $t_1, \cdots, t_m$, which is monotone increasing for every t_k if the $m-1$ other variables are fixed; let $F_k(x)$ $(k=1, \cdots, m)$ be projective functions of the class A_n (or C_n), defined on R. Then the composite function $G(F_1, \cdots, F_m)$ is also projective of the class A_n (or C_n). (II) Let $G(t_1, \cdots, t_m)$ be a Baire function of $t_1, \cdots, t_m$, and let $F_k(x)$ $(k=1, \cdots, m)$ be projective functions of the class B_n, defined on R. Then the composite function $G(F_1, \cdots, F_m)$ is also projective of the class B_n. *A. Rosenthal* (Albuquerque, N. M.).

Kunugui, Kinjiro. Sur une propriété des ensembles plans de mesure positive. Proc. Imp. Acad. Tokyo **17**, 461–465 (1941). [MF 14727]

A set M in the plane is said to be rectangular if M is the Cartesian product of a set A on the x-axis and a set B on the y-axis. The author proves that every set of positive measure on the plane contains a rectangular perfect set. This answers negatively a problem of the reviewer. *P. Erdős* (Stanford University, Calif.).

Tsuji, Masatsugu. Some metrical theorems on a set of points. Proc. Imp. Acad. Tokyo **19**, 110–113 (1943). [MF 14799]

Given an n-dimensional space and a vector r in that space, $E+r$ denotes the set obtained by applying the translation r to the set E. All sets are measurable and mE denotes the measure of E. The paper proves (1) $mE(E+r) \to mE$ as $r \to 0$; (2) and (3) two similar theorems; (4) and (5) two theorems of Steinhaus [Fund. Math. **1**, 93–104 (1920)]; (6) if $mE_1=mE_2$ then decompositions $E_1=e_{10}+\sum e_{1n}$, $E_2=e_{20}+\sum e_{2n}$ can be found such that $me_{10}=me_{20}=0$, while e_{2n} is a translation of e_{1n}. *H. D. Ursell* (Leeds).

Tsuji, Masatsugu. On the uniform distribution of values of a function mod. 1. Proc. Imp. Acad. Tokyo **19**, 66–69 (1943). [MF 14795]

Let $f(x)$ be a real continuous function defined for $0 \leqq x < \infty$ and $(f(x))=f(x)-[f(x)]$, so that $0 \leqq (f(x)) < 1$. Let $r>0$, $0 \leqq \alpha < \beta \leqq 1$, and denote by $E(r, \alpha, \beta)$ the set of points x satisfying the conditions (1) $0 \leqq x \leqq r$, (2) $\alpha \leqq (f(x)) \leqq \beta$. Denote its measure by $mE(r, \alpha, \beta)$. The values of $f(x)$ are said to be distributed uniformly mod 1 if for any α, β we have $\lim r^{-1}mE(r, \alpha, \beta)=\beta-\alpha$ as $r \to \infty$. The following theorem is established. If $f(x)$ is a positive continuous increasing convex function of $\log x$ such that $\lim f(x)/\log x=\infty$ as $x \to \infty$, then the values of $f(x)$ are distributed uniformly mod 1. The concept of uniform distribution mod 1 is also extended to a set of m functions of n variables by replacing the interval (1) by the sphere $x_1^2+\cdots+x_n^2 \leqq r^2$ and (2) by a parallelepiped inside the unit cube of m-dimensional space. It is then shown that a set of m polynomials in n variables, having no constant (nontrivial) linear combination, have values which are distributed uniformly mod 1. *I. J. Schoenberg* (Philadelphia, Pa.).

Linés Escardó, E. Note on the mean measure of unbounded sets. Revista Mat. Hisp.-Amer. (4) **4**, 131–137 (1944). (Spanish) [MF 12167]

A definition of mean measure of unbounded linear point sets given by the author in an earlier paper [same Revista (4) **1**, 43–51 (1941); these Rev. **3**, 75] is made precise and generalized to n-dimensional point sets. *H. S. Wall.*

Moran, P. A. P. Additive functions of intervals and Hausdorff measure. Proc. Cambridge Philos. Soc. **42**, 15–23 (1946). [MF 14407]

Let h be a real continuous increasing function of a real variable such that $h(0)=0$. The Hausdorff measure h-mE of a set E in Euclidean space of q dimensions is defined as $\lim_{\epsilon \to 0} h$-$m_\epsilon E$, where h-$m_\epsilon E$ is the greatest lower bound of numbers of the form $\sum h(d_i)$, where $d_i < \epsilon$ is the diameter of U_i and $\sum U_i \supset E$. This measure function is used here, as by Hausdorff, in the investigation of sets of fractional dimension. The notation p-mE is used for h-mE whenever $h(t)=t^p$. Theorem: if E is a closed bounded set and $E=E_1+\cdots+E_n$, where the E_i are closed nonoverlapping sets similar geomet-

rically to E but reduced in the ratio t_i, then the dimension of E is p_0 and $0<p_0\text{-}mE<\infty$, where p_0 is the root of the equation $t_1^p+\cdots+t_n^p=1$. A more general theorem is proved. The basic result is a criterion that $h\text{-}mE$ is positive in terms of an additive function of half-open figures.

Concerning product sets, the author proves that, if A is any measurable p-dimensional set $(0<p\leqq 1)$ on the OX axis such that $0<p\text{-}mA<\infty$ and B is any measurable linear set on the OY axis such that $0<|B|<\infty$, where $|B|$ is the linear measure of B, then there exist constants K_1, K_2 such that

$$0<K_1\leqq\frac{(1+p)\text{-}m(A\times B)}{(p\text{-}mA)\times|B|}\leqq K_2<\infty,$$

where $A\times B$ is the Cartesian product of A and B. The natural extension to allow B to have dimension $q<1$ is stated to be not possible, as shown by an example in a forthcoming paper by Besicovitch and the author.

J. F. Randolph (Oberlin, Ohio).

Ugaheri, Tadasi. On the Newtonian capacity and the linear measure. Proc. Imp. Acad. Tokyo **18**, 602–605 (1942). [MF 14785]

The paper proves that a bounded set E in the plane, of finite linear measure in the sense of Hausdorff, is of Newtonian capacity zero, Newtonian capacity being defined with reference to the Newtonian potential $1/r$.

H. D. Ursell (Leeds).

Kakeya, Sôichi, Mibu, Masamiti, and Yonemura, Hideo. Relation between the measures $\Lambda_n(X)$ and $m^*(X)$. Proc. Imp. Acad. Tokyo **19**, 241–243 (1943). [MF 14817]

It is shown that the Lebesgue measure and the n-dimensional Hausdorff measure of any subset of Euclidean n-space are equal. A. Sard [Bull. Amer. Math. Soc. **49**, 758–759 (1943); these Rev. **5**, 62] has given a similar proof.

H. Federer (Providence, R. I.).

Denjoy, Arnaud. La mesure euclidienne. C. R. Acad. Sci. Paris **221**, 477–480 (1945). [MF 14688]

The author remarks on two basic principles of measure theory, ascribed respectively to Eudoxos and Jordan.

L. H. Loomis (Cambridge, Mass.).

Kakutani, Shizuo, und Kodaira, Kunihiko. Über das Haarsche Mass in der lokal bikompakten Gruppe. Proc. Imp. Acad. Tokyo **20**, 444–450 (1944). [MF 14911]

Let Ω be a locally bicompact space which is the union of a denumerable number of bicompact sets. By definition a (real) function is called a Baire function if it belongs to the smallest ring of functions on Ω which contains the continuous functions and is closed under the operation of taking pointwise limits of denumerable sequences. A Baire set in Ω is one whose characteristic function is a Baire function. A Carathéodory outer measure μ^* in Ω is called a Lebesgue measure of the first kind if (i) every Borel set is measurable; (ii) bicompact sets have finite outer measure; (iii) for every $A\subset\Omega$, $\mu^*(A)=\inf\mu^*(U)$, where U varies over all open coverings of A. A Carathéodory outer measure m^* in Ω satisfying the above conditions with "Borel," "open," replaced by "Baire," "open Baire," respectively, in (i) and (iii) is called a Lebesgue measure of the second kind. There is a one-to-one correspondence between such measures and the μ^* corresponding to a given m^* is an extension of m^*. If Ω is a group and m^* is a left invariant measure of the second kind it coincides with the corresponding μ^*. The proof is based upon the following principle: every statement concerning at most a denumerable number of Baire functions which is valid for every separable, metric, locally compact space is valid also for Ω.

N. Dunford.

Kakutani, Shizuo. Construction of a non-separable extension of the Lebesgue measure space. Proc. Imp. Acad. Tokyo **20**, 115–119 (1944). [MF 14876]

It is shown that in the unit interval there exists a countably additive measure $m(A)$ which is an extension of Lebesgue measure and is such that the space L_2 of functions of integrable square with respect to it has dimension 2^c, $c=2^{\aleph_0}$, dimension being defined as the cardinal number of a complete orthonormal set. Another property, equivalent to this, is that, if $\mathfrak{B}$ denotes the Borel field on which m is defined, and $\mathfrak{A}$ is any subfamily of $\mathfrak{B}$ with the property that for any $B\in\mathfrak{B}$ and any $\epsilon>0$ there exists an $A\in\mathfrak{A}$ such that $m(A\Delta B)<\epsilon$, then the cardinal number of $\mathfrak{A}$ is 2^c. The construction is based on a lemma of Tarski [Fund. Math. **32**, 45–63 (1939), Hilfssatz 3.16] and requires both the axiom of choice and the continuum hypothesis.

J. C. Oxtoby (Bryn Mawr, Pa.).

Kakutani, Shizuo. Notes on infinite product measure spaces. I. Proc. Imp. Acad. Tokyo **19**, 148–151 (1943). [MF 14807]

A comparatively elementary proof is given for the existence of an independent product measure on an infinite Cartesian product space.

N. Dunford.

Kakutani, Shizuo. Notes on infinite product measure spaces. II. Proc. Imp. Acad. Tokyo **19**, 184–188 (1943). [MF 14811]

[Cf. the preceding review.] In defining an independent product measure on an infinite Cartesian product of compact Hausdorff spaces the customary method leads to a measure defined on the Borel field determined by the system of neighborhoods used to define the weak product topology. In general this Borel field is a proper subset of the field of all Borel sets in the product space. In the present note it is shown that the usual product measure admits an extension to a countably additive measure defined for all Borel subsets of the product space.

N. Dunford.

Mibu, Yoshimiti. Relations between measure and topology in some Boolean space. Proc. Imp. Acad. Tokyo **20**, 454–458 (1944). [MF 14913]

Let Ω be a bicompact Hausdorff space in which the closure of any open set is open and the family $\mathfrak{E}$ of closed open sets constitutes a base. In the class $\mathfrak{E}$ let a finitely additive measure be given which vanishes only for the void set and is such that $m(\Omega)=1$. Suppose also that for any increasing sequence of closed open sets E_n we have $\lim m(E_n)=m(\sum E_n)$. It is shown that the outer measure $m^*A=\inf m(E)$, $E\supset A$, defines a completely additive extension of m such that a set has measure zero if and only if it is nowhere dense (or of first category), and a set is measurable if and only if it has the property of Baire. The possibility of satisfying the hypotheses in a space devoid of isolated points is not discussed.

J. C. Oxtoby (Bryn Mawr, Pa.).

Buch, Kai Rander. Some investigations of the set of values of measures in abstract space. Danske Vid. Selsk. Math.-Fys. Medd. **21**, no. 9, 70 pp. (1 plate) (1945). [MF 15401]

The author investigates the properties of the set of values taken on by those functions of sets known as measures which are defined in a Borel ring in an abstract space. Let

$\mathfrak{B}_1$ denote the usual Borel ring of sets on the interval $0 \leqq x < 1$ and let $\mathfrak{F}$ denote a Borel ring of sets in an abstract space. In parts III and IV of the paper the following theorems are established. (1) The set of numbers whose elements are the values of a bounded measure φ defined in $\mathfrak{B}_1$ is closed. (2) The set of numbers whose elements are the values of a bounded measure ψ defined in $\mathfrak{F}$ is closed. (3) The set of points $[\varphi(A), \psi(A)]$ in the plane, where φ and ψ are bounded measures defined for sets A of $\mathfrak{B}_1$, is a closed set. (4) The set of points $[\varphi(A), \psi(A)]$ in the plane, where φ and ψ are bounded measures defined for sets A in $\mathfrak{F}$, is a closed set.

Theorem (1) is proved by using (i) a representation of φ in terms of a nondecreasing function on $0 \leqq x \leqq 1$ and (ii) a theorem on series of positive terms which is proved at the beginning of part III. The proof of theorem (2) is obtained by mapping the sets of the Borel ring $\mathfrak{F}$ onto the sets of the Borel ring $\mathfrak{B}_1$. The proof of theorem (3) employs the representations of the measures φ and ψ by means of nondecreasing functions on $0 \leqq x \leqq 1$ and the decomposition of such functions into their continuous and discontinuous components. It is shown that the theorem is true if it is true in the following two special cases: (i) the case in which the two nondecreasing functions are continuous; (ii) the case in which the variation of each nondecreasing function is equal to its total discontinuity on each interval. The special case (ii) is proved easily by means of an application of the theorem on series of positive terms, but the proof of special case (i) requires a series of complicated constructions. Finally, the proof of theorem (4) is obtained by mapping the sets of $\mathfrak{F}$ onto the sets of $\mathfrak{B}_1$ as before.

Parts I and II of the paper lay the necessary foundations for the proofs of the results in parts III and IV. Part I gives necessary preliminary definitions and then treats the following topics. (i) The theory of measures and contents in an abstract space. Necessary and sufficient conditions that a content must satisfy in order that it can be extended to a measure are given. (ii) The definite integral of nonnegative functions defined in an abstract space. (iii) Absolutely continuous and singular functions of a set, including the Lebesgue and Jordan decompositions of such functions. (iv) The indefinite integral, including the Radon-Nikodym theorem which states necessary and sufficient conditions for the representation of a function of a set as an indefinite integral. Part II continues with a treatment of the following topics. (v) The decomposition of a nondecreasing function into continuous and discontinuous components. (vi) Functions of a set having the Borel class of sets on the axis of real numbers as the defining region. The author establishes the relations between measures defined in the Borel class and nondecreasing functions. (vii) Functions of a set having $\mathfrak{B}_1$ as defining region. The nondecreasing function corresponding to a measure on $\mathfrak{B}_1$ can be decomposed into continuous and discontinuous components. Each component yields a measure on $\mathfrak{B}_1$. The values taken on by the measure function arising from the continuous component form a closed interval, and in part III it is proved that those taken on by the measure function arising from the discontinuous component form a closed set.

G. B. Price (Lawrence, Kan.).

Wolff, J. Simple proof of the Arzela-Osgood theorem of the integral calculus. Mathematica, Zutphen. B. 10, 90–93 (1941). (Dutch) [MF 15734]

The theorem considered is the following. If the functions $f_n(x)$, Riemann-integrable in the nonempty interval $a \leqq x \leqq b$, are uniformly bounded on (a, b) and $f_n(x) \to 0$ for every x on this interval, then $\lim \int_a^b f_n(x)dx = 0$. The proof is made by means of a lemma which asserts that, if W_n $(n = 1, 2, \cdots)$ is a monotone decreasing sequence of unions of sets, the measure of each set being greater than p (>0), then there is at least one point which belongs to every W_n. *A. Dresden.*

Doubrovsky, V. On some properties of completely additive set functions and passing to the limit under the integral sign. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 311–320 (1945). (Russian. English summary) [MF 15361]

The author proves the following theorem. Let M be a Borel field of subsets of a set E, let f_n be a uniformly bounded sequence of M-measurable functions such that $f_n(x)$ converges to a limit $f(x)$ for every x in E, and let Φ_n be a sequence of completely additive set functions defined on M such that $\Phi_n(e)$ converges to $\Phi(e)$ for every e in M; then Φ is completely additive and

$$\lim_{n\to\infty} \int_E f_n \Phi_n(de) = \int_E f\Phi(de),$$

where the integrals are taken in the Lebesgue-Stieltjes sense. *M. M. Day* (Urbana, Ill.).

Bosanquet, L. S. Some properties of Cesàro-Lebesgue integrals. Proc. London Math. Soc. (2) 49, 40–62 (1945). [MF 13708]

The function $g(t)$ has a $C_\lambda L$ integral in $(0, a)$, denoted by $C_\lambda L \int_0^a g(t)dt$, if (i) $g(t)$ is summable in (ϵ, a), $0 < \epsilon < a$, and (ii) there exists, for $\lambda = 0$, $\lambda > 0$, respectively,

$$\lim_{\epsilon\to 0} \int_\epsilon^a g(t)dt, \quad \lim_{\epsilon\to 0} (\lambda/\epsilon^\lambda) C_\mu L \int_0^\epsilon (\epsilon - t)^{\lambda-1} dt \int_t^a g(u)du,$$

$$\mu = \max(\lambda - 1, 0).$$

If $g(t) \in C_\lambda L$ in $(0, a)$, then $g(t) \in C_{\lambda'} L$ in $(0, a)$ for $\lambda' > \lambda$ and the integrals are equal. Hence, if $g(t) \in C_\lambda L$ for some unspecified λ the integral is independent of λ. It is denoted by $C\int_0^t g(t)dt$, $g(t) \in (CL)$. The following relations between the (CL) integral and the Cesàro-Perron integral of Burkill hold. If $g(t) \in C_\lambda L$ then $g(t) \in C_\lambda P$ and the integrals are equal; if (i) $g(t) \in L$ in (ϵ, a), $0 < \epsilon < a$, (ii) $g(t) \in C_\lambda P$ in $(0, a)$, then $g(t) \in C_\lambda L$ in $(0, a)$.

For $t > 0$ define

$$G_\alpha(t) = \frac{1}{\Gamma(\alpha)} C \int_0^t (t-u)^{\alpha-1} g(u)du, \qquad \alpha > 0,$$

$G_0(t) = g(t)$, and, for $\alpha \geqq 0$, $g_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} G_\alpha(t)$. Here the function $(t-u)^{\alpha-1} g(u) \in C_\lambda L$ in $(0, t)$; $G_\alpha(t)$ is unique. If (i) $g(t) \in C_\mu L$ in $(0, a)$ and is $o(t^p)$ (C, λ) as $t \to +0$, $p > -1$, (ii) $h(t)$ is defined in $(0, a)$, $h^{(\rho)}(t) = O(t^{-\tau-\rho})$ in $(0, a)$ for $\rho = 0, 1, \cdots, m$, $\tau < p+1$ and m the least integer such that $m \geqq \lambda$, then $g(t)h(t) \in C_\mu L$ in $(0, a)$, where $\mu = \max(\lambda - 1, 0)$, and is $o(t^{p-\tau})(C, \lambda)$ as $t \to +0$.

If (i) $g(t) \in C_\lambda L$ in $(0, a)$, (ii) $G_1(t) = o(t^p)$ (C, λ) as $t \to +0$, where $p \geqq 0$, (iii) $h(t)$ is defined in $(0, a)$ and such that

$$C \int_0^a t^{m+\tau} |dh^{(m)}(t)| < \infty,$$

m an integer, then $g(t)h(t) \in C_\lambda L$ and

$$C_\lambda L \int_0^a g(t)h(t)dt = G_1(a)h(a) - C_\mu L \int_0^a G_1(t)h'(t)dt$$

if $\lambda>0$, while if $\lambda<0$

$$CL\int_0^a g(t)h(t)dt=G_1(a)h(a)-CL\int_0^a G_1(t)dh(t).$$

If $g(t)\varepsilon C_\lambda L$ in $(0, a)$ and $\alpha>0$, then, for $0\leqq t\leqq a$,

$$G_{\alpha+1}(t)=\frac{1}{\Gamma(\alpha)}C_\alpha L\int_0^t(t-u)^{\alpha-1}G_1(u)du.$$

If $\alpha\geqq 0$, $G_{\alpha+1}(t)$ is AC in every interval (ϵ, a), $0<\epsilon<a$, and $(d/dt)G_{\alpha+1}(t)=G_\alpha(t)$.

If $g(t)\varepsilon(CL)$ in $(0, a)$, a necessary and sufficient condition that $g(t)\varepsilon C_\lambda L$ in $(0, a)$ is that $G_{\alpha+1}(t)=o(t^\alpha)$ $(C, \lambda-\alpha)$ as $t\to+0$, where $-1<\alpha\leqq\lambda$; a necessary and sufficient condition that $g(t)=o(t^p)$ (C, λ) as $t\to+0$, where $p>-1$, $\lambda\geqq 0$, is that $g_\alpha(t)=o(t^p)$ $(C, \lambda-\alpha)$ as $t\to+0$, where $0\leqq\alpha\leqq\lambda$.

For $t>0$, let $g(t)\varepsilon L$ for every interval (ϵ, Δ), $0<\epsilon<\Delta<\infty$; write $\xi(t)=t^{\sigma-1}\int_t^\infty u^{-\sigma}g(u)du$, the integral being assumed convergent. The following results then hold. (i) If $\xi(t)\varepsilon C_\lambda L$ in $(0, a)$, then $g(t)\varepsilon C_{\lambda+1}L$ in $(0, a)$. (ii) If $\sigma\neq 0$ and $g(t)\varepsilon C_{\lambda+1}L$ in $(0, a)$, there is a number s such that $\xi(t)-st^{\sigma-1}\varepsilon C_\lambda L$ in $(0, a)$. If $\alpha>0$, s is arbitrary. If $\alpha>0$ and $g(t)\varepsilon(CL)$ in $(0, \Delta)$ for $\Delta>0$, then $\xi(t)\varepsilon(CL)$ in $(0, \Delta)$ and for $\alpha\geqq 0$, $t>0$, $\xi_\alpha(t)=t^{\sigma-1}\int_t^\infty u^{-\sigma}g_\alpha(u)du$.

If $\alpha>0$ and $g(t)\varepsilon(CL)$ in $(0, \Delta)$ for $\Delta>0$, a necessary and sufficient condition that $g(t)/t^\sigma\varepsilon C_\lambda L$ and $C_\lambda L\int_0^\infty u^{-\sigma}g(u)du=l$ is that $g_\alpha(t)/t^\sigma\varepsilon C_{\lambda-\alpha}L$ and, for $0\leqq\alpha\leqq\lambda$,

$$C_{\lambda-\alpha}L\int_0^\infty u^{-\sigma}g_\alpha(u)du=\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(\sigma+\alpha)}l.$$

R. L. Jeffery (Kingston, Ont.).

Areškini, G. On the theory of multiple integrals in abstract spaces. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 357–363 (1944). (Georgian and Russian) [MF 14603]

The author considers a system R_k, $1\leqq k\leqq n$, of spaces with continuous mappings f_k, $1\leqq k\leqq n-1$, of R_k onto R_{k+1}. For each point x^{k+1} in R_{k+1} the set $X(x^{k+1})$ is the set of all x^k in R_k for which $f_k(x^k)=x^{k+1}$. Let $\mu(x^{k+1})$ be a measure function defined on Borel subsets of $X(x^{k+1})$. If F^k is a function defined on R_k and integrable with respect to $\mu(x^{k+1})$ on each $X(x^{k+1})$, then setting

$$F^{k+1}(x^{k+1})=\int_{X(x^{k+1})}F^k d\mu(x^{k+1})$$

defines a function F^{k+1} on R_{k+1}. Let μ_n be a measure on R_n. Then, under certain conditions on the spaces R_k and certain integrability conditions on the functions occurring during the process, the iterated integral

$$\int_{R_n}\left\{\int_{X(x^n)}\left\{\cdots\left\{\int_{X(x^2)}F^1 d\mu(x^2)\right\}\cdots\right\}d\mu(x^n)\right\}d\mu_n$$

exists and can be regarded as $\int_{R_1}F^1 d\mu$ for suitable μ. Similar considerations with locally compact topological groups are said to lead to results on Haar measures. Details are to appear later. *M. M. Day* (Urbana, Ill.).

Corominas, Ernesto. Differential properties of continuous functions which lack angular points. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 41–62 (1945). (Spanish) [MF 15369]

The author observes that many of the classical properties of derivatives can be generalized to the Dini derivatives of continuous functions which need not be differentiable but have no corners, a corner being a point at which the upper derivative on one side is less than the lower derivative on the other side. For example, the law of the mean holds for one (at least) of the two derivatives on either side. One (at least) of the derivatives on either side takes all values between its upper and lower bounds. Any of the derivatives omits at most a countable number of values between its upper and lower bounds; at most two derivatives, one from each side, one upper and one lower, can omit even one value. *R. P. Boas, Jr.* (Providence, R. I.).

Izumi, Shin-ichi, and Sunouchi, Gen-ichirô. Notes on differentiation. Proc. Imp. Acad. Tokyo **19**, 105–109 (1943). [MF 14798]

The authors first establish a number of theorems on the differentiability of indefinite integrals and on related problems. The proofs employ a theorem of Kantorovitch on the convergence of a sequence of operations from one regular vector lattice to another [see C. R. (Doklady) Acad. Sci. URSS (N.S.) **14**, 255–259 (1937)] and the maximal theorem of Hardy and Littlewood [Acta Math. **54**, 81–116 (1930); Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, p. 150] with extensions by Privaloff and by Takahashi [Sci. Rep. Tôhoku Imp. Univ. **25**, 56–78 (1936)]. The authors first apply their method to prove the standard theorem of Lebesgue theory that a function $x(t)\varepsilon L$ is equal to the derivative of its indefinite integral almost everywhere. The same method is used to prove the theorem of Jessen, Marcinkiewicz and Zygmund on the strong differentiability of $\int\int_I x(s, t)dsdt$ [see Fund. Math. **25**, 217–234 (1935) or Saks, Theory of the Integral, Warsaw-Lwów, 1937, p. 147]. There follow various extensions of the Hardy-Littlewood theorem for functions of several variables. These results include a theorem due to Wiener [Duke Math. J. **5**, 1–18 (1939)]. One consequence is the theorem that an indefinite integral of an integrable function of several variables is regularly differentiable almost everywhere. Finally, the authors prove a convergence theorem for sequences of operations from one regular vector lattice to another which includes theorems of the class of Denjoy's theorem on differentiation and its analogue. *G. B. Price.*

Besicovitch, A. S. On the existence of tangents to rectifiable curves. J. London Math. Soc. **19**, 205–207 (1944). [MF 14565]

The author presents a very short proof of the existence almost everywhere of tangents to a rectifiable curve. The proof is geometric and avoids measure theory. It makes no mention of the imbedding space except to assume the existence locally of straight line segments (or geodesics) and properties of their lengths and angles. It thus holds in any metric space which locally is sufficiently like Euclidean or Hilbert space. *L. H. Loomis* (Cambridge, Mass.).

Besicovitch, A. S. A general form of the covering principle and relative differentiation of additive functions. II. Proc. Cambridge Philos. Soc. **42**, 1–10 (1946). [MF 14405]

The author has previously [same Proc. **41**, 103–110 (1945); these Rev. **7**, 10] extended Vitali's covering theorem for sets in the Euclidean plane to any nonnegative additive function of sets, the covering sets being circles. In this paper the covering sets are more general and he obtains results concerning the differentiation of indefinite integrals. Let $F(X)$ be a nonnegative function of sets X additive over a class of sets which includes all Borel sets. Let $F^*(Y)$

denote the greatest lower bound of $F(X)$ over all B-measurable sets X including Y. A sequence of closed sets X_n is F-regular and convergent to a if (i) diameter$(X_n+a)\to 0$; (ii) for some k and all n, $F(X_n)\geqq kF\{c(a,r)\}$, where $c(a,r)$ is the least circle with center a including X_n. The set Γ of closed sets covers E in the general Vitali sense if, for any a in E, Γ contains an F-regular sequence convergent to a. Theorem: if Γ covers E in the general Vitali sense there is a subset $\bar{\Gamma}$, consisting of a finite or enumerable set of disjoint sets from Γ, such that $F^*(E-\bar{\Gamma})=0$.

Defining a general relative derivative $D_\theta(x, \Theta/F)$ of an additive Θ over F-regular sequences convergent to x, the author proves that this derivative exists at almost all points (in the F-sense) when $r>0$ implies $F\{c(x,r)\}>0$. Finally, the indefinite integral $\int_X f(x)d\mu$ over a set E of finite μ-measure is finitely derivable at almost all points with respect to $\mu(XE)$, and a necessary and sufficient condition that the derivative is almost everywhere $f(x)$ is obtained in terms of a certain (B, μ)-measurability of $f(x)$.

U. S. Haslam-Jones (Oxford).

Besicovitch, A. S. On the definition and value of the area of a surface. Quart. J. Math., Oxford Ser. **16**, 86–102 (1945). [MF 14169]

The point set σ in E^3 which carries a continuous surface $x_i(u_1, u_2)$, $i=1, 2, 3$; $0\leqq u_i\leqq 1$, may have positive 3-dimensional Lebesgue measure, but the Lebesgue area of the surface may be finite [for an old example of Geöcze, in which σ is a cube and the area is 0, cf. T. Radó, On the Problem of Plateau, Ergebnisse der Math., vol. 2, no. 2, Springer, Berlin, 1933, p. 5]. The present paper shows that the same may occur even when the mapping $u\to x(u)$ is topological. [The main idea, namely, the "imbedding" of a Cantor set in E^3 in an infinite polyhedron of finite area, has already been used by Nöbeling, Math. Ann. **118**, 687–701 (1943); these Rev. 6, 44.] The author then modifies his example slightly to yield a surface in E^3 where the u-space is the surface of a sphere, the mapping $u\to x(u)$ is topological, the Lebesgue area is equal to the classical integral and as small as desired, but the volume of the closed set bounded by σ is as large as desired.

These examples lead the author to make a statement to the effect that Lebesgue's definition of area and all results proved concerning it "have to go." The situation seems to the reviewer to be as follows. Rectifiable curves in E^2, say, have these properties: (1) the 2-dimensional Lebesgue measures of their carriers are 0; (2) the isoperimetric problem can be solved for closed sets; (3) arc length is lower semicontinuous and (4) expressible by the classical integral. Radó [op. cit., p. 4] showed that the requirement of lower semicontinuity leads naturally to Lebesgue area. Besicovitch's examples show then that, for area, even if restricted to surfaces where $u\to x(u)$ is topological, either of properties (1) or (2) is incompatible with either of properties (3) or (4). Whether the first two or the last two properties are deemed essential is a matter of taste and depends on whether area is considered primarily as a geometric or as an analytical concept. Besicovitch's statement presumably means that he emphatically endorses the geometric point of view. However, for those who adopt the analytical point of view, the many and deep recent investigations on Lebesgue area will lose none of their significance. [It should be observed that the isoperimetric problem is solvable by Lebesgue area if the volume of the open set bounded by σ is considered as the number which is to be maximized [see L. Tonelli, Rend. Circ. Mat. Palermo **39**, 109–138 (1915)]. Tonelli never states whether he considers the open or closed set with boundary σ, but his proofs consistently use the open set.]

For those who favor the geometric attitude, all results on area will, in the author's words, "have to be done anew." The second part of the paper takes a first step in this direction by proving the following. Let $f(x, y)$ be absolutely continuous in the Tonelli sense in a rectangle R of the (x, y)-plane. If G is a measurable subset of R and E the set of points $(x, y, z=f(x, y))$ with $(x, y)\varepsilon G$, then the two-dimensional Carathéodory measure [usually called Hausdorff measure: defined in terms of the squares of the diameters of covering sets] of E is given by $\int_G(1+f_x^2+f_y^2)^{\frac{1}{2}}dxdy$.

H. Busemann (Northampton, Mass.).

Besicovitch, A. S. On the definition of the area of a surface by means of inscribed polyhedra. J. London Math. Soc. **19**, 138–141 (1944). [MF 13634]

The paper gives an example of a continuous surface $z=f(x, y)$, $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, which presents the following phenomenon. It is possible to divide the unit square into arbitrarily small triangles, of a shape very near to any assigned shape, such that the area of the corresponding inscribed polyhedron is either arbitrarily large or else very nearly equal to the area of the given surface, as desired. This example is intended to supplement the paper reviewed above.

T. Radó (Columbus, Ohio).

Ridder, J. Über das Flächenmasz im dreidimensionalen Raum. Nieuw Arch. Wiskunde (2) **21**, 33–56 (1941). [MF 15711]

Ridder, J. Errata zur Arbeit: Über das Flächenmasz. Nieuw Arch. Wiskunde (2) **21**, 268–269 (1943). [MF 15707]

Throughout most of this paper the author reviews and amplifies the theory of the Burkill integral and its derivatives with respect to arbitrary measures. In conclusion he defines a surface area in terms of Burkill integrals by considering inscribed polyhedra corresponding to regular triangulations of the parameter space and derives some properties of this area from the theory of Burkill integration.

H. Federer (Providence, R. I.).

Radó, Tibor. On continuous mappings of Peano spaces. Trans. Amer. Math. Soc. **58**, 420–454 (1945). [MF 14231]

The reviewer [Amer. J. Math. **57**, 17–50 (1935)] stated a theorem which may be stated in more modern terminology as follows. Let $p^*=T_1(p_1)$ and $p^*=T_2(p_2)$ be two continuous mappings from 2-cells into subsets of Euclidean space. Let $T_i=L_iM_i$ (that is, $T_i(p_i)=L_i\{M_i(p_i)\}$), $i=1, 2$, be monotone light factorizations of T_i having middle spaces Q_1 and Q_2. Then T_1 and T_2 are equivalent in the sense of Fréchet ($T_1\sim T_2(F)$) if and only if there exists a homomorphism h from Q_1 to Q_2 such that $L_2hM_1=L_1$ on Q_1. The "only if" part of this theorem is true but the "if" part has been shown to be false by J. W. T. Youngs [see Ann. of Math. (2) **45**, 753–785 (1944); these Rev. 6, 278]. It appears that the homeomorphism h must possess additional orientation properties which have been discussed by Youngs in the paper referred to above for the case that T_1 and T_2 are 2-spheres. Using the false theorem, the reviewer [Amer. J. Math. **58**, 313–322 (1936)] proved a "cyclic additivity" theorem for the Lebesgue area of surfaces [see below], which was correctly proved by Youngs for surfaces of the topological type of the sphere.

The attempt to extend the results of Youngs to the case of mappings T defined on 2-cells (instead of 2-spheres) intro-

duces many new difficulties. Rather than confining himself to this case, the author has presented an elegant treatment of mappings T from any metric Peano space P to (subsets of) any metric space P^*. He has shown that the equivalence theorem stated above is true if h is also required to satisfy the following condition. If Z_1 is any continuous mapping from Q_1 into a subset of itself then $L_1Z_1M_1 \sim L_2hZ_1h^{-1}M_2$ in the sense of Fréchet.

Now let T be a mapping as above and let $T=LM$ be a monotone light factorization of T with middle space Q. The author defines an (unordered) sequence $(T^1, T^2, \cdots)$ of partial mappings as follows: (1) if Q is a dendrite, the sequence is empty; (2) otherwise, let $Q^1, Q^2, \cdots$ be the proper cyclic elements of Q and let m^n be the unique monotone retraction of Q into Q^n; then $T^n=Lm^nM$. This sequence is shown to be independent of the particular factorization. Now let P and P^* be fixed and let $F(T)$ be any real valued function of mappings T. Then $F(T)$ is defined to be "additive with respect to continua of constancy" if, for every choice of T, a continuum γ over which T is constant, and a set D which is a component of $P-\gamma$, we have $F(T)=F(T_1)+F(T_2)$, where $T_1=T$ in D, $T(\gamma)$ in $P-D$; $T_2=T(\gamma)$ in D, T in $P-D$. The author then generalizes the cyclic additivity theorem for Lebesgue area as follows. (a) Suppose that P possesses the property II (shown by Whyburn to be equivalent to unicoherence, P being a Peano space). (b) Suppose that $0 \leqq F(T) \leqq +\infty$ for all T and that $F(T)$ is lower-semicontinuous (uniform convergence). (c) Suppose also that $F(T)=0$ if T is a constant or Q is an arc and suppose that $F(T)$ is "additive with respect to continua of constancy." Then $F(T)=\sum F(T^n)$. The conditions (c) and several others are also shown to be necessary whether P is unicoherent or not. Many other results are obtained, including a number of results concerning the equivalence theorem for the case that P is a 2-cell or 2-sphere. *C. B. Morrey, Jr.*

Youngs, J. W. T. On surfaces of class K_1. Bull. Amer. Math. Soc. **51**, 669–673 (1945). [MF 13601]

Radó has defined a Fréchet surface (of the type of the disk) of class K_1 as a Fréchet surface which possesses a parametric representation $x^i=x^i(u^1, u^2)$, $i=1, 2, 3$, on a closed Jordan region $\bar{R}$, which is of class K_1. (Such a representation is of class K_1 if each coordinate function is continuous and possesses partial derivatives almost everywhere on the interior R and if the Jacobians $\partial(x^i, x^j)/\partial(u^1, u^2)$ are summable.) Radó has shown that, for such representations, $L(S) \geqq I(x^i)$, $L(S)$ being the Lebesgue area of S and $I(x^i)$ being the classical area integral, the equality holding only if the representation is "of class K_2." The author proves that every Fréchet surface is of class K_1 and, in fact, possesses a representation on the unit square Q: $0 \leqq u^1, u^2 \leqq 1$ in which the x^i are continuous and $\partial x^i/\partial u^1=\partial x^i/\partial u^2=0$ almost everywhere. He gives the following construction of such a representation for any surface S. Let $x^i=\bar{x}^i(\bar{u}^1, \bar{u}^2)$ be any continuous representation of S on Q. Let $s(x)$ denote the Cantor function. Then $x^i=\bar{x}^i[s(u^1), s(u^2)]$ is a representation of the type described above. The author proposes three questions, one of which is: is $L(S)=\sup I(x^i)$ for all possible representations of class K_1 of S? *C. B. Morrey, Jr.*

Menger, Karl. Définition intrinsèque de la notion de chemin. C. R. Acad. Sci. Paris **221**, 739–741 (1945). [MF 15163]

In the coordinate-free theory of length and line integrals developed by the author in earlier papers [for example, Rice Inst. Pamphlet **27**, 1–40 (1940); these Rev. **3**, 229] only the use of the classical definition of path as a class of mappings of a parameter interval into a complete metric space prevented the theory from being completely intrinsic. In the present note the author eliminates this last extrinsic element by giving an intrinsic definition of path. If $f[a, b]$, $g[c, d]$ are motions (continuous transformations of intervals $[a, b]$, $[c, d]$ into a complete metric space) they are equivalent provided a transformation τ of $[a, b]$ onto $[c, d]$ exists such that the points $f(t)$, $g(\tau(t))$ are identical for each point t of $[a, b]$; the classical theory defines a path as a maximal class of equivalent motions. To a set $T=\{t_0, \cdots, t_n\}$, $t_i \varepsilon [a, b]$, $t_i<t_{i+1}$ $(i=0, \cdots, n-1)$, $t_0=a$, $t_n=b$, there corresponds by $f[a, b]$ a chain of closed sets $\{E_1, \cdots, E_n\}$, where E_k is the set of positions corresponding to $t \varepsilon [t_{k-1}, t_k]$. A chain denotes a finite sequence of closed sets containing more than one point (f is assumed not constant on any subinterval of $[a, b]$), each two consecutive elements of the sequence having a nonempty product. Calling $\mathfrak{C}(f)$ the class of chains corresponding to all finite subsets T of $[a, b]$, it is seen that (1) $\mathfrak{C}(f)$ contains chains of arbitrarily small norms (the norm of a chain being the greatest of the diameters of its elements); (2) for each two chains C_1, C_2 of $\mathfrak{C}(f)$ the class contains a common refinement of C_1 and C_2; (3) the class $\mathfrak{C}(f)$ is saturated with respect to property (2). The author defines a path to be a class $\mathfrak{C}$ with properties (1), (2), (3). Each path, in conjunction with a corresponding aggregate of sets contained in $[a, b]$, determines a motion $f[a, b]$ such that $\mathfrak{C}=\mathfrak{C}(f)$, and two motions f, g determined by the same path are necessarily equivalent. The determination of $f[a, b]$ can be based upon any sequence $C_1, C_2, \cdots$ of chains of class $\mathfrak{C}$ provided that each chain C_{k+1} is a refinement of chain C_k and the norms of C_k approach zero. Consequently each such sequence constitutes another definition of path (two sequences determine the same path whenever they belong to the same saturated class $\mathfrak{C}$). If $\mathfrak{P}$ denotes the class of polygons $\{P\}$, $P=\{f(t_0), \cdots, f(t_n)\}$, $\{t_0, \cdots, t_n\}$ a finite subset of $[a, b]$, then for each two polygons P_1, P_2 the class contains a polygon common to P_1, P_2, $\mathfrak{P}$ is saturated with respect to this property and $\mathfrak{P}$ contains polygons whose norms (with respect to $\mathfrak{P}$) are arbitrarily small. The author gives in conclusion a definition of path as a class $\mathfrak{P}$ and states that this definition furnishes a satisfactory basis for an intrinsic theory of path length. *L. M. Blumenthal.*

Tsuji, Masatsugu. On Green's lemma. Proc. Imp. Acad. Tokyo **18**, 176–178 (1942). [MF 14751]

The author proves the following extension of Green's lemma. Let Γ be a rectifiable Jordan curve and D its interior. Let $A(x, y)$ be continuous and bounded in D and such that $\lim A(x, y)$ exists almost everywhere on Γ as $(x, y) \to \Gamma$ nontangentially. For almost all y_0 let $A(x, y_0)$ be absolutely continuous on the intersection of D and $y=y_0$, and let $\partial A/\partial x$ be summable over D. Then

$$\int\int_D (\partial A/\partial x)dxdy = \int_\Gamma A dy.$$

In proof the author maps $D \cup J$ conformally onto the closed unit circle and applies the known theory of such mappings. *L. H. Loomis* (Cambridge, Mass.).

Theory of Functions of Complex Variables

***Valiron, Georges. Théorie des Fonctions.** Masson et Cie., Paris, 1942. ii+522 pp.

[This is part I of the author's Cours d'Analyse Mathématique.] The object of this book is to start with the basic

facts about the real number system and to develop a fairly complete rigorous account of the calculus and of the classical theory of functions of a single complex variable. The theory is developed under hypotheses which are restrictive enough to avoid unduly lengthy technical arguments and are at the same time sufficiently general to be useful in a wide variety of applications. The Lebesgue integral appears in only one chapter of twelve pages, where a succinct account is given of its definition and fundamental properties. However, this chapter has a supplementary character and the Lebesgue integral is not used elsewhere in the book. Each chapter is preceded by a brief historical account of the subject matter treated. Noteworthy features of the book are: (1) a careful detailed account of multiple integral theory under the usual classical assumptions; (2) a brief treatment of continued fractions, a topic not customarily treated in such texts; (3) a proof of the Hadamard theorem concerning the asymptotic behavior of $\pi(x)$, the number of primes not exceeding x; (4) contact with modern research in analysis by frequent reference to recent monographs dealing with special topics.

In the part concerned with the theory of functions of a complex variable emphasis is placed upon geometric methods and the theory of conformal mapping. In the words of the author ". . . j'ai amorcé l'étude de quelques théories dont l'exposition, grâce aux efforts de nombreux mathématiciens contemporains, se présente désormais sous une forme simple et n'exigeant de larges développements." In this connection the fundamental theorem concerning the one-to-one conformal mapping of simply-connected regions is used as a starting point to develop the theory of the Schwarz mapping functions and the elliptic modular function. The classical theorems of Picard, Landau, Schottky, and Julia are then developed.

An idea of the scope of the book may be had from the following list of chapter headings. I. Numbers, sets, limits. This chapter includes a brief account of continued fractions and statements, without proof, of certain topological theorems. II. Infinite series and products. III. Generalities on continuous functions. Functions of a single variable. IV. The Riemann integral. Applications and extensions. This chapter includes proofs of the transcendental character of e and π as well as a brief account of the Stieltjes integral. V. The Lebesgue integral. VI. Functions of a single complex variable defined or represented by series or integrals. VII. Trigonometric series and generalizations. VIII. Reduction and mechanical computation of integrals. IX. Functions of several variables. X. Double integrals. XI. Triple and multiple integrals. XII. Elementary functions of a complex variable. XIII. The Cauchy theory. Fundamental theorems and the calculus of residues. XIV. The Weierstrass theory. Analytic continuation. XV. Conformal mapping. XVI. Elliptic functions. XVII. Analytic functions defined by integrals. Applications.

M. H. Heins (Cambridge, Mass.).

Minami, Unai. On the Cauchy's integral theorem. Proc. Imp. Acad. Tokyo **18**, 440–445 (1942). [MF 14770]

The author gives another proof of the so-called strong form of Cauchy's theorem by approximating the rectifiable boundary curve from within by polygons.

L. H. Loomis (Cambridge, Mass.).

Catunda, Omar. On a modification of Cauchy's formula. Summa Brasil. Math. **1**, 9–16 (1946). (Portuguese. English summary) [MF 15864]

Let C be an oriented contour in the complex plane and call its interior the set of points which are to the left when C is described in the positive sense; thus ∞ may be inside C. Then the formula

$$f(z)=(2\pi i)^{-1}\int_C f(t)\frac{(\lambda-z)dt}{(\lambda-t)(t-z)}$$

is valid if $f(z)$ is analytic in the interior of C and continuous in the closed interior, z is inside C and λ is outside. The author observes that this formula is invariant under linear fractional transformations and derives similar expressions which represent linear combinations of $f(z)$ and its derivatives.

R. P. Boas, Jr. (Providence, R. I.).

Steinberg, N. Sur les conditions suffisantes pour qu'une fonction soit monogène. Rec. Math. [Mat. Sbornik] N.S. **17(59)**, 45–58 (1945). (Russian. French summary) [MF 14592]

Menchoff [same Rec. N.S. **2(44)**, 339–356 (1937)] has shown that, if $f(z)$ is continuous in a domain D and if for each z_0 in D, except for at most a denumerable set of points of D, there is a family of circles $\Gamma_n(z_0): |z-z_0|=\rho_n$ with

$$(*)\qquad \lim_{n\to\infty}\rho_n=0,\quad \lim_{n\to\infty}R_n'/r_n'=1,$$

where R_n' and r_n' denote the greatest and least distances of the curve $\Gamma_n'=f[\Gamma_n]$ from the point $z_0'=f(z_0)$, then $f(z)$ is an analytic function of z or of $\bar{z}$. In the first part of the present paper, the author weakens the hypothesis (*), showing that the function $f(z)$, continuous in D, is an analytic function of z or of $\bar{z}$ in D provided (a) for each z_0 of D there is a sequence of circles with

$$\lim_{n\to\infty}\rho_n=0,\quad \limsup_{n\to\infty}v_n'/d_n'^2=\pi/4,$$

where v_n' and d_n' are the area and diameter, respectively, of the finite domain bounded by Γ_n' and (b) for each z_0 of $D-A$ there is a sequence of circles with

$$\lim_{n\to\infty}\rho_n=0,\quad \liminf_{n\to\infty}\rho_{n+1}/\rho_n>0,\quad \liminf_{n\to\infty}v_n'/d_n'^2>0,$$

where the set A has measure zero, and, for each line $x=x_0$ and each line $y=y_0$, excepting at most a set of values of x_0 and y_0 of linear measure zero, the intersection of A with the line is at most a denumerable set of points.

In the second part of the paper the author considers a class of functions $f(z)$ defined on a nowhere dense perfect set C, and for them establishes analogues of Morera's theorem and of theorems of Rademacher and Menchoff on conditions of analyticity.

E. F. Beckenbach.

Kametani, Shunji. On conditions for a function to be regular. Jap. J. Math. **17**, 337–345 (1941). [MF 14962]

Several theorems are proved, related to that of Besicovitch, each deducing the regularity of a function of one complex variable from its continuity everywhere and its satisfying almost everywhere additional properties related to the difference quotient but more complicated than the existence of a derivative.

P. Franklin.

Martin, Yves. Sur l'extension à certaines séries de quelques propriétés des séries entières. C. R. Acad. Sci. Paris **219**, 385–387 (1944). [MF 15270]

The author considers series of the type $f(z)=\sum_{n=0}^{\infty}c_n\varphi(z,n)$. The $\varphi(z,n)$ are entire functions such that $0<m<|\varphi(z,n)/z^n|<M$ as $n\to\infty$ for any $0<r<|z|<R<\infty$. These series behave like power series when they have a finite radius of convergence. They converge uniformly and absolutely for $|z|\leqq\rho'<\rho$ with $\rho^{-1}=\limsup|c_n|^{1/n}$ and diverge for $|z|>\rho$.

On the circle of convergence there is the usual Abel summability and the corresponding Tauberian theorem.

P. Civin (Buffalo, N. Y.).

Martin, Yves. Sur une classe de développements en série. C. R. Acad. Sci. Paris **219**, 666–668 (1944). [MF 15300]

This is a continuation of the work described in the preceding review. The series is now considered as representing an entire function. The author shows that the order of the function does not depend on the magnitude of the $|c_n|$. If the further hypothesis is made that the $\varphi(z, n)$ are polynomials of degree at most n, then the preceding statement is no longer true. *P. Civin* (Buffalo, N. Y.).

Marth, Ella. Further properties of Garvin's *F*-series. Duke Math. J. **12**, 645–653 (1945). [MF 15507]

The *F*-series (a generalized Lambert series) is

$$F(z)=\sum_{n=1}^{\infty} a_n z^{n\lambda}(1-z^{n\mu})^{-1},$$

where λ and μ are integers [Garvin, Amer. J. Math. **58**, 507–513 (1936)]. By evaluating $\lim (1-z/z_0)F(z)$ as $z\to z_0$ (a rational boundary point) along a radius, Garvin showed that under suitable conditions the (unit) circle of convergence is a natural boundary for $F(z)$. The author evaluates the same limit for approach through an angle. She also shows that, if $\sum|a_n|$ converges and z_0 is an irrational boundary point, then the limit is zero for radial approach and, for some irrational points, also for approach in an angle.

R. P. Boas, Jr. (Providence, R. I.).

Broggi, Ugo. Sulle funzioni determinanti regolari all'infinito. Boll. Un. Mat. Ital. (2) **4**, 12–15 (1942). [MF 16050]

Mambriani, Antonio, e Mambriani, Silvia. Sulle singolarità delle funzioni analitiche definite da integrali di Laplace. Boll. Un. Mat. Ital. (2) **4**, 235–238 (1942). [MF 16074]

Two direct proofs of the fact, apparently first proved by Pincherle, that a function represented by a Laplace integral and regular at infinity has a singular point on its axis of convergence. *R. P. Boas, Jr.* (Providence, R. I.).

Schaginjan, A. Remarques sur l'étude de l'approximation par les fonctions rationnelles dans un domaine complexe. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 11–14 (1945). [MF 15226]

Continuing the work of earlier papers [for example, same C. R. (N.S.) **27**, 318–320 (1940); **44**, 47–51 (1944); these Rev. **2**, 188; **6**, 149], theorems are stated on weighted mean uniform approximations to analytic functions by polynomials in domains whose complements are not connected. For an example of the type of theorem, see the review of second reference above. Proofs are not given, but it is stated that the method is that used previously [same C. R. (N.S.) **45**, 50–52 (1944); these Rev. **7**, 64]. Misprints in the second reference above are listed. [There is an obvious misprint in the exponent in the last line of page 11.]

I. M. Sheffer (State College, Pa.).

Keldych, M. Sur la représentation par des séries de polynomes des fonctions d'une variable complexe dans de domaines fermés. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 249–258 (1945). (Russian. French summary) [MF 14581]

The author gives a proof of the following theorem which generalizes his earlier result [same Rec. N.S. **8(50)**, 137–148 (1940); these Rev. **2**, 188]; the case of a Jordan region was considered by Walsh [Math. Ann. **96**, 430–436 (1927)]. If $\bar{D}$ is a closed region whose complement is a single region containing the point at infinity, then every function continuous in $\bar{D}$ and analytic in its interior can be represented in $\bar{D}$ by a uniformly convergent series of polynomials. Since the condition on $\bar{D}$ is easily seen to be necessary, this is the best possible result of its kind. *R. P. Boas, Jr.*

Tsuji, Masatsugu. On the zeros of the Riemann zeta-function. Proc. Imp. Acad. Tokyo **18**, 631–634 (1942). [MF 14790]

Another proof of Littlewood's theorem that $\zeta(s)$ $(s=\sigma+it)$ has a zero in every strip $|t-T|<A/\log\log\log T$ $(T\geqq T_0)$, depending on Doetsch's three-lines theorem [Math. Z. **8**, 237–240 (1920)]. *E. C. Titchmarsh* (Oxford).

Gilly, Jean. Sur une extension de la théorie de la composition de première espèce. C. R. Acad. Sci. Paris **218**, 100–102 (1944). [MF 13456]

This note considers the extension of the composition operation to certain classes of functions of negative infinite order. These are of the form

$$K(x, y)=(y-x)^{\alpha}\sum_{0}^{\infty}((y-x)^{-N}/\Gamma(\alpha-N+1))k_N(x, y).$$

Then, following Hadamard's idea, the finite part of

$$\int_0^b (b-x)^{\alpha}A(x)dx$$

(where b is a singular point for $A(x)$) is

$$\lim_{\epsilon\to 0}\int_a^{b-\epsilon}(b-x)^{\alpha}A(x)dx+\epsilon^{\alpha}B(b-\epsilon),$$

where $B(x)$ has a singularity at $x=b$ and is chosen so that the limit exists. For uniqueness purposes, ϵ is taken complex and the curve joining a and $b-\epsilon$ is chosen to avoid the branch cuts. [For functions of the closed cycle, some of the main ideas occur in L. M. Blumenthal, Amer. J. Math. **53**, 483–492 (1931), but no reference is made to this paper.]

D. G. Bourgin (Urbana, Ill.).

Gilly, Jean. Comparaison entre la théorie de la composition et la transformation de Laplace-Carson. C. R. Acad. Sci. Paris **218**, 382–384 (1944). [MF 13467]

Let $M(x, y)=m(x+\mu(x); y)$, $N(x, y)=n(x; y-\nu(y))$, where the functions vanish for $y-x<\mu(x)$, $\mu>0$ and $y-x<\nu(y)$, $\nu>0$, respectively. The author states the relation for composition of the first kind, namely

$$M^*N=m^*n(x+\mu(x); y-\nu(y))$$

or $M^*N=0$ according as $y-x$ exceeds or is inferior to $\mu(x)+\nu(y)$. There is also a collection of statements about the relation between the Laplace transform and the formal operational equivalent of a function, $f(t)=\sum_{k=1}^{\infty}c_kt^{\alpha_k}/\Gamma(\alpha_k+1)$. Most of this last material is hardly new.

D. G. Bourgin (Urbana, Ill.).

Gilly, Jean. Étude analytique des produits de composition. C. R. Acad. Sci. Paris **219**, 383–385 (1944). [MF 15269]

The author considers functions $f(z)$, $g(z)$ which are multiplied, respectively, by $e^{2\pi i\alpha}$, $e^{2\pi i\beta}$ when z describes certain contours surrounding singularities of the functions. He then defines the composition operation by

$$(e^{2\pi i\alpha}-1)^{-1}(e^{2\pi i\beta}-1)^{-1}\int_C f(u)g(t-u)du,$$

where C is a double contour like those occurring in Pochhammer's integrals for the gamma function. This generalized composition has the proper formal properties.

R. P. Boas, Jr. (Providence, R. I.).

Bergman, Stefan. A remark on the mapping of multiply-connected domains. Amer. J. Math. **68**, 20–28 (1946). [MF 15486]

Let B be bounded by the regular closed curves $b_0, b_1, \cdots, b_p$; b_0 is the outer contour. The author operates with a complete orthogonal system generated by the functions 1, z^k, $(z-\alpha_\mu)^{-k}$ $(\mu=1, \cdots, p; k=1, 2, \cdots)$, where the point α_μ is enclosed by b_μ. If the functions of this system are denoted by ϕ_k', it is readily seen that, for example, those analytic functions w in B whose real part is single-valued and constant on each contour can be developed in a series $\sum c_k\phi_k$, where the integrals ϕ_k are defined in a cut-up region. The author remarks that this method often yields geometric inequalities. As an example he derives an estimate for the distortion $|w'|$ in the case considered above.

L. Ahlfors (Zurich).

Lammel, Ernst. Anwendung der Funktionentheorie auf die Theorie der Potentialströmungen um von Kreisen gebildete Profile. Monatsh. Math. Phys. **51**, 24–34 (1943). [MF 14379]

The author obtains the most general potential flow exterior to two disjoint circles which has these circles as streamlines and has a prescribed velocity at infinity. The single-circle solution is also obtained. *L. H. Loomis.*

Komatu, Yûsaku. Einige Anwendungen der Verzerrungssätze auf Hydrodynamik. Proc. Imp. Acad. Tokyo **19**, 454–461 (1943). [MF 14842]

The classical distortion theorems associated with the conformal mapping of the exterior of the unit circle onto the exterior of a bounded continuum are interpreted by the author as inequalities on the magnitude and direction of the velocity of an ideal incompressible fluid flowing past the continuum. The circulation about the boundary is not assumed to be zero. The paper concludes with a proof of the following inequality for the real part of the coefficient b_1 of $1/z$ in the series expansion of the mapping function:

$$\frac{A/\pi}{1+(1-4A/\pi\delta^2)^{\frac{1}{2}}} \leqq \Re b_1 \leqq \frac{A/\pi}{1-(1-4A/\pi\delta^2)^{\frac{1}{2}}},$$

where A is the area of the continuum (and its interior) and δ is its diameter. *L. H. Loomis* (Cambridge, Mass.).

Komatu, Yûsaku. Über Verzerrungen bei der konformen Parallelschlitzabbildung von zweifach zusammenhängenden Gebieten. Proc. Imp. Acad. Tokyo **21**, 1–5 (1945). [MF 14948]

Let G be a doubly-connected region in the z-plane containing the point at infinity, and let Γ_0 and Γ_1 be its boundary components. Let $w=f(z)$, normed by $f(\infty)=\infty$, $f'(\infty)=1$, map G one-to-one and conformally onto a "slit" region of the w-plane, the two boundary segments being parallel to the real axis and having imaginary parts β_0 and β_1, respectively. The author proves that if M_i and m_i are, respectively, the maximum and minimum values of $y(=\Im(z))$ on Γ_i then $m_1-M_0 \leqq \beta_1-\beta_0 \leqq M_1-m_0$, with equality only in case G is itself a slit region of the same kind. Moreover, if $w=g(z)$ is any single-valued analytic function which is regular on G except for a pole at infinity, with $f'(\infty)=1$, and if K_i and k_i are, respectively, the limit superior and limit inferior of $\Im(g(z))$ as z approaches Γ_i, then the mapping function $w=f(z)$ can be characterized to within an additive constant as the solution of the extremal problem of maximizing k_1-K_0, or minimizing K_1-k_0. *L. H. Loomis.*

Komatu, Yûsaku. Die Geschwindigkeitspotentiale und die Kutta-Joukowskischen Bedingungen für die Strömungen in vielfach zusammenhängenden Gebieten. I. Proc. Imp. Acad. Tokyo **21**, 6–15 (1945). [MF 14949]

Let G be a multiply-connected region bounded by n continua Γ_i and containing the point at infinity in its interior. In terms of the mapping function for G (onto a "slit" region) and a generalized Green's function for G, the author constructs the velocity potential for the unique flow in G having a given velocity at infinity and given circulations about the boundary components Γ_i. If the velocity at infinity and the points of separation on the boundary components Γ_i are prescribed, then the circulation constants are dependent, and the author gives a formula for determining them.

L. H. Loomis (Cambridge, Mass.).

Komatu, Yûsaku. Die Geschwindigkeitspotentiale und die Kutta-Joukowskischen Bedingungen für die Strömungen in vielfach zusammenhängenden Gebieten. II. Proc. Imp. Acad. Tokyo **21**, 83–93 (1945). [MF 14952]

The considerations of the paper reviewed above are applied to the case of a circular annulus and the various functions and relations are explicitly calculated in terms of elliptic functions. *L. H. Loomis* (Cambridge, Mass.).

Komatu, Yûsaku. Sur la représentation de Villat pour les fonctions analytiques définies dans un anneau circulaire concentrique. Proc. Imp. Acad. Tokyo **21**, 94–96 (1945). [MF 14953]

Villat's formula for the solution of the Dirichlet problem for a circular annulus [Rend. Circ. Mat. Palermo **33**, 134–174 (1912)] is derived from the Green's function of the annulus. *L. H. Loomis* (Cambridge, Mass.).

Komatu, Yûsaku. Über eine Verschärfung des Löwnerschen Hilfssatzes. Proc. Imp. Acad. Tokyo **18**, 354–359 (1942). [MF 14768]

Let $w=f(z)$ map the unit circle $|z|<1$ with $f(0)=0$, $f'(0)>0$, into a domain (not necessarily schlicht) lying inside the unit circle $|w|<1$, and suppose that an arc of length l of the circle $|z|=1$ is mapped continuously on an arc of length L of the circle $|w|=1$. H. Unkelbach [Math. Z. **43**, 739–742 (1938)] has given the inequality $L \geqq 2l/(1+f'(0))$, a strengthened version of the lemma used by Löwner to establish his well-known differential equation. The author considers only schlicht mappings and, by integrating Löwner's equation, proves for these mappings the stronger inequality $\sin L/4 \geqq f'(0)^{\frac{1}{2}} \sin l/4$. There is equality only if $f(z)$ is the slit mapping defined by the equation $f(z)/(1+\epsilon f(z))^2 = f'(0)z/(1+\epsilon z)^2$, where $-\bar{\epsilon}$ is the midpoint of the arc of length l on $|z|=1$. An application is given.

D. C. Spencer (Stanford University, Calif.).

Komatu, Yûsaku. Sur la variation d'une fonction de représentation conforme, lorsque le domain varie. Proc. Imp. Acad. Tokyo **19**, 599–608 (1943). [MF 14855]

Let D be a domain in the w-plane bounded by a simple closed analytic curve C and let a be a fixed interior point of D. Let $z=\phi(w)$, $\phi(a)=0$, $\phi'(a)>0$ be the function which maps D on the unit circle $|z|<1$. Various authors, including G. Julia [C. R. Acad. Sci. Paris **172**, 568–570, 738–741, 831–833 (1921); Ann. Sci. École Norm. Sup. (3) **39**, 1–28

(1922)], have determined the first order variation of the mapping function $z=\phi(w)$ when the curve C is slightly varied. The author gives a proof of Julia's formula based on the Poisson integral representation of a function harmonic in the unit circle [cf. A. C. Schaeffer and the reviewer, Duke Math. J. **10**, 611–635 (1943); **12**, 107–125 (1945); these Rev. **5**, 175; **6**, 206] and generalizes Julia's formula to doubly-connected domains. *D. C. Spencer.*

Komatu, Yûsaku. Einige Darstellungen analytischer Funktionen und ihre Anwendungen auf konforme Abbildung. Proc. Imp. Acad. Tokyo **20**, 536–541 (1944). [MF 14920]

Let $f(z)$ be regular for $|z|<1$ and piecewise regular on $|z|=1$, the exceptional points being $z=e^{i\phi_\mu}$, $\mu=1,\cdots,m$; $0<\phi_1<\cdots<\phi_m\leqq 2\pi$, and at these points suppose that $\lim f'(e^{i\phi})$ exists as $\phi\to\phi_\mu\pm 0$, and that $f'(z)$ does not vanish on $|z|=1$ at points other than the exceptional points. Using the Poisson-Stieltjes representation of Herglotz, the author obtains the representation

$$f''(z)/f'(z)=\pi^{-1}\int_0^{2\pi}(e^{i\phi}-z)^{-1}d\arg f'(e^{i\phi}).$$

Applying this representation to the class of functions $w=f(z)$, $f(0)=0$, $f'(0)=1$ which are schlicht for $|z|<1$ and which map the unit circle on convex domains, the author obtains the well-known representation

$$1+zf''(z)/f'(z)=(2\pi)^{-1}\int_0^{2\pi}\{(e^{i\phi}+z)/(e^{i\phi}-z)\}d\rho(\phi),$$

from which known properties of the functions $f(z)$ may be obtained.

A further application is made in obtaining the Schwarz-Christoffel formula for the conformal mapping of the unit circle on polygonal domains. *M. S. Robertson.*

Komatu, Yûsaku. Darstellungen der in einem Kreisringe analytischen Funktionen nebst den Anwendungen auf konforme Abbildung über Polygonalringgebiete. Jap. J. Math. **19**, 203–215 (1945). [MF 15003]

Let $f(z)$ be single-valued and regular in the ring $q<|z|<1$ and piecewise regular on the boundary $q\leqq|z|\leqq 1$, the exceptional points being $z=e^{i\sigma_\mu}$, $z=qe^{i\psi_\nu}$. At these points suppose that $\lim f'(e^{i\phi})$ exists for $\phi\to\phi_\mu-0$ and $\phi\to\phi_\mu+0$ and that $\lim f'(qe^{i\phi})$ exists for $\phi\to\psi_\nu-0$ and $\phi\to\psi_\nu+0$. Suppose that $f'(z)$ does not vanish in $q\leqq|z|\leqq 1$ at points other than the exceptional points. Using the Villat-Stieltjes representation and the Weierstrass elliptic functions the author obtains

$$\frac{zf''(z)}{f'(z)}=\frac{1}{\pi i}\int_0^{2\pi}\zeta(i\log z+\phi)d\arg f(e^{i\phi})-\frac{1}{\pi i}\int_0^{2\pi}\zeta_1(i\log z+\phi)d\arg f'(qe^{i\phi})+ic^*,$$

where

$$\pi^2c^*=\eta_1\left\{\int_0^{2\pi}\phi d\arg f'(e^{i\phi})-\int_0^{2\pi}\phi d\arg f'(qe^{i\phi})\right\}.$$

An application is made to the conformal mapping of a circular ring domain onto a polygonal ring domain.

M. S. Robertson (New Brunswick, N. J.).

Kveselava, D. On the conformal mapping of adjacent regions. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **5**, 463–472 (1944). (Georgian and Russian) [MF 14605]

Let the function $w=f(z)$ map conformally a simply-connected domain G in the z-plane onto the circle $|w|<1$, and let t denote a boundary point of G. If the limit (finite or infinite) $f'(t)=\lim_{z\to t}\{f(z)-f(t)\}/(z-t)$ exists as z approaches t along any path in G not touching the boundary, the point t is called "normal." Two simply-connected domains G and G_1 bounded by curves γ and γ_1 are said to be adjacent if they have no interior points in common and if their boundaries have an arc γ_0 in common (if γ and γ_1 coincide, G and G_1 are complementary). The author obtains upper bounds for the product $|f'(t)f_1'(t)|$ at normal points of γ_0, where f and f_1 are functions mapping G and G_1 on $|w|<1$, and also gives an upper bound for $|f'(t)|$ at normal points of a domain G at which G has a supporting line (normal points of convexity). *D. C. Spencer.*

Denjoy, Arnaud. Sur la représentation conforme. C. R. Acad. Sci. Paris **219**, 11–14 (1944). [MF 14515]

This note is concerned with functions $f(z)$ which are defined, analytic, and bounded for $|z|<1$ and in addition are such that the Riemannian image of $|z|<1$ with respect to $f(z)$ has a finite area. The principal theorem established is that, if E is the subset of $|z|=1$ of measure 2π at which the Fatou radial limits of $f(z)$ exist (these limits being denoted by $f(a)$ if $a\in E$), then the subset of E which contains those a for which

$$\int_E|\{f(\zeta)-f(a)\}/\{\zeta-a\}|^2|d\zeta|<+\infty$$

is of measure 2π. This theorem is deduced from an identity relating the area of the Riemannian image of $|z|\leqq r(<1)$ with respect to $f(z)$ and the integral

$$\int_\gamma|dz_1|\int_\gamma|\{f(z_2)-f(z_1)\}/\{z_2-z_1\}|^2|dz_2|,$$

γ being the circle $|z|=r$. Applications are indicated to problems concerning the behavior of $f(z)$ and $f'(z)$ as z tends to a point a in a specified manner. *M. H. Heins.*

Huron, Roger. Sur deux lemmes de représentation conforme. C. R. Acad. Sci. Paris **221**, 367–369 (1945). [MF 14675]

The author gives a new proof of a lemma of Kravtchenko [J. Math. Pures Appl. (9) **20**, 35–234, 235–303 (1941), in particular, p. 136; these Rev. **3**, 219; **4**, 58] concerning the boundary behavior of a conformal mapping function. The present proof establishes the lemma under less restrictive hypotheses than those of Kravtchenko and eliminates the use of the elliptic modular function. *M. H. Heins.*

Haruki, Hiroshi. Two theorems on the univalent function. Proc. Phys.-Math. Soc. Japan (3) **25**, 622–623 (1943). [MF 15074]

Let $f(z)=z+\sum_2^\infty a_nz^n$ be regular and satisfy $a|z|\leqq|f(z)|\leqq b$ in $|z|<1$. The author shows that $|zf'(z)/f(z)-1|<1$ for $|z|<a/b$, which implies that $f(z)$ is univalent and starforming with respect to the origin for $|z|<a/b$. In the second theorem $f(z)=\sum_0^\infty a_nz^n$ is assumed regular for $|z|\leqq\rho$ $(\rho>1)$. Let

$$M_n=\max_{|z|=\rho}\left|f(z)-\sum_{k=0}^{n}a_kz^k\right|,\quad K_n=\max_{2\leq k\leq n}|a_k|.$$

If, for some positive integer n,

$$|a_1|>\tfrac{1}{2}(n-1)(n+2)K_n+\rho(\rho-1)^{-2}M_n,$$

then $f(z)$ is univalent for $|z|\leqq 1$.

M. S. Robertson (New Brunswick, N. J.).

Yosida, Tokunosuke. Bemerkungen über die p-wertigen Funktionen. Proc. Imp. Acad. Tokyo **20**, 16–19 (1944). [MF 14868]

Let $w(z)=z+\sum_0^\infty a_n z^{-n}$ be regular for $\infty>|z|>1$ and $(w(z))^p$ multivalent of order p for $|z|>1$. Then $\sum_{n=1}^\infty n|a_n|^2\leqq 1$. The author's theorem is an extension of a similar theorem for the case $p=1$ due to L. Bieberbach. In addition, he shows that, with the same hypothesis, $w(z)$ is univalent for $|z|>2^{\frac{1}{2}}$.

Let $f(z)=z^p+a_{p+1}z^{p+1}+\cdots+a_nz^n+\cdots$ be regular and multivalent of order p for $|z|<1$. Then $|a_{p+1}|\leqq 2p$, $|f(z)|\geqq 2^{-p/2}|z|^p(1+|z|)^{-2p}$ for $|z|<1$, and

$$(f(z))^{1/p}=z+b_2z^2+\cdots+b_nz^n+\cdots$$

is regular and univalent for $|z|<2^{-\frac{1}{2}}$. *M. S. Robertson.*

Yosida, Tokunosuke. Ein Satz über die p-wertigen Funktionen. Proc. Imp. Acad. Tokyo **20**, 409 (1944). [MF 14905]

Let $f(z)=z^p+a_{p+1}z^{p+1}+\cdots+a_nz^n+\cdots$ be regular and multivalent of order p for $|z|<1$. In the paper reviewed above the author showed that for $|z|<1$

$$|f(z)|\geqq A(p)|z|^p(1+|z|)^{-2p},$$

where $A(p)=2^{-p/2}$. In this note he obtains the larger estimate $(1.0617)^{-p}$ for $A(p)$. *M. S. Robertson.*

Kobori, Akira. Sur les fonctions multivalentes. Proc. Imp. Acad. Tokyo **20**, 216–217 (1944). [MF 14886]

Let $f(z)=z^p+a_{p+1}z^{p+1}+\cdots+a_nz^n+\cdots$ be regular and multivalent of order p for $|z|<1$. Then for $|z|\leqq\tanh 1=0.761\cdots$ the author obtains

$$|f(z)|\leqq(1+|z|^{\frac{1}{2}})^{2p}|z|^p(1-|z|)^{-2p}.$$

M. S. Robertson (New Brunswick, N. J.).

Joh, Kenzo, and Hukusima, Yutaka. On the "Verzerrungssatz" of p-valent functions. Proc. Phys.-Math. Soc. Japan (3) **25**, 377–383 (1943). [MF 15058]

Let $f(z)=z^p+a_{p+1}z^{p+1}+\cdots+a_nz^n+\cdots$ be regular and multivalent of order p for $|z|\leqq 1$. Using Ahlfors's theory of conformal representation, M. L. Cartwright [Math. Ann. 111, 98–118 (1935)] showed that

$$|f(z)|\leqq A(p)r^p(1-r)^{-2p},\qquad |z|=r<1,$$

and that $A(p)\leqq e^{3+p}$ [misprinted in the author's paper]. Given a ring domain G in the z-plane we may represent G on a domain $r<|w|<R$ by an analytic function $w(z)$. Let $M=\log R/r$. O. Teichmüller [Deutsche Math. **3**, 621–678 (1938)] has shown that if G' is a ring domain contained in G then $M'\leqq M$. The authors employ this theorem to show that $A(p)\leqq 16^p$, an improvement upon Cartwright's estimate. *M. S. Robertson* (New Brunswick, N. J.).

Kametani, Shunji, and Ugaheri, Tadasi. A remark on Kawakami's extension of Löwner's lemma. Proc. Imp. Acad. Tokyo **18**, 14–15 (1942). [MF 14737]

Soit dans le domaine circulaire unité D de frontière Γ $f(z)$ holomorphe telle que $|f(z)|<1$, $f(0)=0$. Soit sur Γ un ensemble E où f admet une limite radiale ρ de module 1. Les points ρ forment un ensemble E' sur Γ. Les auteurs montrent, par usage des résultats de Fatou [Acta Math. **30**, 335–400 (1906)] sur l'intégrale de Poisson, que la mesure intérieure de E est majorée par la mesure extérieure de E'. *M. Brelot* (Grenoble).

Tsuji, Masatsugu. On an extension of Löwner's theorem. Proc. Imp. Acad. Tokyo **18**, 220–221 (1942). [MF 14755]

Reprenant l'énoncé du théorème de Löwner étendu selon Kametani et Ugaheri [voir l'analyse précédente], l'auteur montre de plus que si E est l'ensemble (maximum) où f admet une limite radiale de module 1, E et E' sont mesurables. On voit d'abord que E est borélien, puis que E' est analytique donc mesurable. Enfin on complète l'inégalité des mesures par l'inégalité stricte si $0<\text{mes } E<2\pi$. *M. Brelot* (Grenoble).

Guelfer, S. On a property of bounded functions. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 291–294 (1945). (Russian. English summary) [MF 14584]

Let E be the class of functions $f(z)$ which satisfy the three conditions of the Schwarz lemma, namely (i) $f(z)$ is regular in $|z|<1$, (ii) $|f(z)|\leqq 1$ in $|z|<1$, (iii) $f(0)=0$. Write $r_k=((1+k^2)^{\frac{1}{2}}-1)/k$. The author proves the following two results, the first of which was proved by Carathéodory in the case $n=1$. (1) If $f(z)=\sum_{n=1}^\infty a_nz^n$ belongs to E, then for $n=1, 2, \cdots, k$ the nth derivative $f^{(n)}(z)$ satisfies $|f^{(n)}(z)|\leqq n!$ in $|z|\leqq r_nr_{n-1}\cdots r_1$ (the inequality is false for $n>k$). (2) If $f(z)=\sum_{n=1}^\infty a_nz^n$ belongs to E, then for $s=1, 2, \cdots, k$ the function $w=f^{(s-1)}(z)$ maps $|z|<r_kr_{k-1}\cdots r_{k-s+1}|a_k|$ onto a domain which is star-like with respect to the point $w=0$. *D. C. Spencer.*

Shimizu, Tatsujiro. On some property of regular functions in $|z|<1$. Proc. Imp. Acad. Tokyo **18**, 140–143 (1942). [MF 14747]

Let $f(z)$ be regular for $|z|<1$ and define, for $\epsilon>0$,

$$M_\theta(r,\epsilon)=\text{l.u.b.}_{\theta-\epsilon<\phi<\theta+\epsilon}|f(re^{i\phi})|,$$

$$\bar{M}_\theta(1)=\text{g.l.b.}_{0<\epsilon<\theta}\limsup_{r\to 1}M_\theta(r,\epsilon).$$

Let E_θ be a set of θ everywhere dense in $(0, 2\pi)$. Then $f(z)$ is said to have the F-property if $\lim_{r\to 1}f(re^{i\theta})$ exists, finite or infinite, for each fixed θ of E_θ; $f(z)$ is said to have the M-property if $\bar{M}_\theta(1)=\infty$ for all θ of E_θ. If $f(z)$ has both the F- and M-properties then the Riemann surface of the inverse function of $f(z)$ has no boundary points in the finite plane. *M. S. Robertson* (New Brunswick, N. J.).

Tumura, Yosiro. Sur les singularités non directement critiques. Proc. Imp. Acad. Tokyo **18**, 606–611 (1942). [MF 14786]

Theorems concerning asymptotic values of entire functions can be extended to meromorphic functions, provided that the number of poles is small. The author states a number of such theorems and promises proofs. *L. Ahlfors* (Zurich).

Tsuji, Masatsugu. Nevanlinna's fundamental theorems and Ahlfors' theorem on the number of asymptotic values. Jap. J. Math. **18**, 675–708 (1943). [MF 14979]

The function $w=f(z)$ is meromorphic in and on the finite boundary of a region Δ extending to ∞. It maps Δ into a circle $[f(z), w_0]<\delta$ of the Riemann sphere and it is supposed that $[f(z), w_0]=\delta$ at all finite boundary points. The distribution of values in Δ is studied, mainly by methods of the reviewer [Acta Math. **65**, 157–194 (1935)]. It is also possible to study the asymptotic values in Δ. The results, which cannot be briefly stated, are analogous to, but not straightforward generalizations of, those valid in the classical case of functions meromorphic in the whole plane. *L. Ahlfors* (Zurich).

Kunugui, Kinjiro. Une généralisation des théorèmes de MM. Picard-Nevanlinna sur les fonctions méromorphes. Proc. Imp. Acad. Tokyo 17, 283–288 (1941). [MF 14707]

Let $w=f(z)$ be meromorphic in an arbitrary region D and let $S_{z_0}^{(D)}$ be the cluster set at a point z_0 of the boundary Γ, that is, the set of all values which are approximated by $f(z)$ in every neighborhood of z_0. There is a corresponding cluster set $S_{z_0}^{(\Gamma)}$ along the boundary which may be defined as the lower topological limit of $S_{z'}^{(D)}$ as $z'\neq z_0$ tends to z_0. Obviously, $S_{z_0}^{(D)}\supset S_{z_0}^{(\Gamma)}$. The author has shown that $S_{z_0}^{(D)}-S_{z_0}^{(\Gamma)}$ is an open set [same Proc. 15, 27–32 (1939)]. Let Ω_n be a component of this set and Δ a subregion of Ω_n which is bounded by a finite number of analytic contours. The author wants to investigate the distribution of the values of $f(z)$ in a component of $f^{-1}(\Delta)$. Taking $z_0=\infty$, the distribution can be characterized by the number of roots of $f(z)=w_0$ between two circles $|z|=\rho_0$ and $|z|=r$ and by a quantity $m(r, w_0)$ which is defined as in Nevanlinna's theory. A characteristic function $T(r, w_0)=N(r, w_0)+m(r, w_0)$ is introduced. It is stated without proof that the order of $T(r, w_0)$ does not depend on w_0, but only on the component Ω_n. This idea is elaborated to include the consideration of disks (islands and half-islands). There are practically no proofs.

L. Ahlfors (Zurich).

Kunugui, Kinjiro. Sur la théorie de la distribution des valeurs. Proc. Imp. Acad. Tokyo 18, 269–275 (1942). [MF 14760]

The same problem as in the preceding paper. The author now presents a proof for an adaptation of Nevanlinna's first theorem. The reviewer had difficulties in following the involved proof, but he does not doubt the result.

L. Ahlfors (Zurich).

Kunugui, Kinjiro. Sur la théorie des fonctions méromorphes et uniformes. Jap. J. Math. 18, 583–614 (1943). [MF 14974]

This paper covers practically the same material as the two communications reviewed above.

L. Ahlfors (Zurich).

Delange, Hubert. Sur les suites de fonctions entières de genre borné et à zéros réels et négatifs. C. R. Acad. Sci. Paris 221, 603–605 (1945). [MF 15146]

Let

$$f_n(z)=e^{a_0+a_1z+\cdots+c_pz^p}\prod(1+z/\alpha_j)\exp\left[-\frac{z}{\alpha_j}+\cdots+(-1)^p\frac{z^p}{p\alpha_j^p}\right],$$

with $c_q=a_q+ib_q$, $\alpha_j>0$, where the constants depend upon n. Define $\nu_n(t)$ as the number of zeros of $f_n(z)$ of modulus not exceeding t, E the set of limit points of the zeros of $f_n(z)$, $-l$ the maximum abscissa of the points of E, $-L$ the upper limit of the zeros, Π the domain made up of the plane cut along the real axis from $-\infty$ to $-L$ and $\phi(n)$ a positive function of the integer n. A sequence of increasing functions $\psi_n(t)$ defined on a certain interval, finite or infinite, is said to be vaguely convergent on this interval to an increasing function $\psi(t)$ if $\psi_n(t)$ tends to $\psi(t)$ for all the values of t for which $\psi(t)$ is continuous and at the ends of the interval when they are finite. Let the sequence $\mu_n(t)=\nu_n(t)/\phi(n)$ be vaguely convergent to $\mu(t)$ and suppose $A>0$ is a value of t for which $\mu(t)$ is continuous. Suppose that

$$\left[c_0+\sum_{\alpha_j\leq A}\log\alpha_j^{-1}\right]\Big/\phi(n)$$

has a limit C_0, that $\{c_q+(-1)^q\binom{p}{q}^{-1}\sum_{\alpha_j\leq A}\alpha_j^{-q}\}/\phi(n)$ has a limit C_q, that $\sum_{\alpha_j>A}\alpha_j^{-p-1}/\phi(n)$ has a limit k. Then $\{1/\phi(n)\}\log f_n(z)$ converges in Π to

$$\int_{-0}^{A}\log(z+t)d\mu(t)+\int_A^{+\infty}\left\{\log\left(1+\frac{z}{t}\right)-\frac{z}{t}+\cdots+(-1)^p\frac{z^p}{pt^p}\right\}d\mu(t)+c_0+c_1z+\cdots+c_pz^p+\frac{(-1)^p}{p+1}\left\{k-\int_A^{+\infty}t^{-p-1}d\mu(t)\right\}z^{p+1}.$$

M. S. Robertson (New Brunswick, N. J.).

Delange, Hubert. Sur les suites de fonctions méromorphes d'ordre borné à zéros et pôles réels et négatifs. C. R. Acad. Sci. Paris 221, 741–743 (1945). [MF 15164]

Let

$$f_n(z)=e^{a_0+a_1z+\cdots+c_pz^p}G(z;\alpha)/G(z;\beta),$$

$$G(z;\alpha)\equiv\prod\left(1+\frac{z}{\alpha_j}\right)\exp\left\{-\frac{z}{\alpha_j}+\cdots+(-1)^p\frac{z^p}{p\alpha_j^p}\right\},$$

$\alpha_j>0$, $\beta_j>0$, $c_q=a_q+ib_q$, the constants depending upon n, except for p which is fixed. Define $\nu_n(t)$ as the number of zeros not less than $-t$ of $f_n(z)$ minus the number of poles not less than $-t$, E the set of limit points of zeros or of poles of $f_n(z)$, $-l$ the maximum abscissa of the points of E, $-L$ the upper limit of the zeros and poles, Π the domain made up of the plane cut along the real axis from $-\infty$ to $-L$ and $\phi(n)$ a positive function of the integer n. Suppose that $M_n(t)=\{1/\phi(n)\}\int_0^t\nu_n(u)du$ "converges vaguely" [see the preceding review] to $M(t)$, that for $t=A>0$, a point of continuity of $M(t)$, $\{c_q+(-1)^q\int_0^t\nu_n(t)t^{-q-1}dt\}/\phi(n)$ has a limit C_q $(q=0, \cdots, p)$ and that $\{1/\phi(n)\}\int_A^\infty\nu_n(t)t^{-p-2}dt$ has a limit k. Then $\{1/\phi(n)\}\log f_n(z)$ converges in Π to

$$-\int_{-0}^{A}\frac{dM(t)}{z+t}-\int_A^{+\infty}\left\{\frac{1}{z+t}-\frac{1}{t}+\frac{z}{t^2}+\cdots+(-1)^{p+1}\frac{z^p}{t^{p+1}}\right\}dM(t)+c_0+c_1z+\cdots+c_pz^p+(-1)^p\left\{k-\int_A^{+\infty}t^{-p-2}dM(t)\right\}z^{p+1}.$$

M. S. Robertson (New Brunswick, N. J.).

Dufresnoy, Jacques. Théorie nouvelle des familles complexes normales. Applications à l'étude des fonctions algébroïdes. Ann. Sci. École Norm. Sup. (3) 61, 1–44 (1944). [MF 14644]

The theory of normal families is applied to systems of $n+1$ functions considered as projective coordinates in an n-dimensional space. The corresponding distance is introduced in the manner of H. and J. Weyl [Ann. of Math. (2) 39, 516–538 (1938)], which makes for a slight but significant difference as compared with Montel's classical theory. The usual necessary and sufficient conditions in terms of equicontinuity and the boundedness of the invariant derivative are proved in detail. For families of systems of analytic functions the basic test is the following: if $2n+1$ linear combinations $a_0f_0+\cdots+a_nf_n$, any $n+1$ of which are linearly independent, do not vanish at any point and for any system $(f_0, \cdots, f_n)$ of the family, then the family is normal. The proof depends on Borel's classical theorem [Acta Math. 20, 357–396 (1897)]. There are various generalizations of this test, including results on algebroid functions. The methods involve no essentially new element, but they are skilfully used to yield formerly unproved, if predictable, results.

L. Ahlfors (Zurich).

Oka, Kiyosi. Sur les fonctions analytiques de plusieurs variables. VI. Domaines pseudoconvexes. Tôhoku Math. J. **49**, 15–52 (1942). [MF 14694]

[Part V appeared in Jap. J. Math. **17**, 523–531 (1941); these Rev. **3**, 85.] Die vorliegende Arbeit gibt die vollständige Lösung eines der wichtigsten Probleme der Theorie analytischer Funktionen zweier komplexen Veränderlichen, indem nachgewiesen wird, dass jeder schlichte und endliche pseudokonvexe Bereich ein Regularitätsbereich ist. Die Pseudokonvexität wird wie folgt definiert. Ein endlicher schlichter Bereich $\mathfrak{D}$ des Raumes der zwei komplexen Veränderlichen x, y heisse pseudokonvex, falls die zu $\mathfrak{D}$ komplementäre Menge E in jedem ihrer Randpunkte (ξ, η) dem Hartogs'schen Kontinuitätssatz genügt; d.h. genauer: immer dann, wenn die Ebene $x=\xi$ in der Umgebung von (ξ, η) ausser (ξ, η) selbst keinen Punkt von E enthält, gebe es zu jedem $\rho>0$ ein $r>0$ derart, dass einem beliebigen x' aus $|x-\xi|<r$ stets ein y' aus $|y-\eta|<\rho$ zugeordnet werden kann, sodass (x', y') der Menge E angehört. Ferner soll diese Eigenschaft gegenüber jeder in (ξ, η) analytischen Abbildung invariant sein. Der Verfasser beweist zunächst einen mit dem ersten Cousinschen Problem eng zusammenhängenden Hilfssatz und leitet dann hieraus den folgenden fundamentalen Satz ab. Wird ein schlichter, beschränkter Bereich $\mathfrak{D}$ von zwei Hyperebenen $x_2=a_1$, $x_2=a_2$, $a_1<a_2$ (x_2 der Imaginärteil von x), so aufgeteilt, dass sowohl der in $x_2<a_2$ wie der in $x_2>a_1$ gelegene Teil von $\mathfrak{D}$ Regularitätsbereiche sind, dann ist auch $\mathfrak{D}$ selbst ein Regularitätsbereich. Es folgt fast unmittelbar: gibt es zu jedem endlichen Randpunkte eines schlichten endlichen Bereiches $\mathfrak{D}$ eine Hyperkugel, deren Durchschnitt mit $\mathfrak{D}$ ein Regularitätsbereich ist (d.h. ist $\mathfrak{D}$ im Cartanschen Sinne pseudokonvex), so ist $\mathfrak{D}$ ein Regularitätsbereich. Es sei nun $\mathfrak{D}$ ein im obigen Sinne pseudokonvexer Bereich und $d(x, y)$ die gewöhnliche Randdistanz eines Punktes (x, y) aus $\mathfrak{D}$. Dann ist $-\log d(x, y)$ eine in $\mathfrak{D}$ "pseudokonvexe" Funktion (ein der subharmonischen Funktion analoger Begriff). Es folgt, dass $\mathfrak{D}$ durch im Cartanschen Sinne pseudokonvexe Bereiche, d.h. also durch Regularitätsbereiche approximiert werden kann, und hieraus nach einem Satze von Behnke-Stein, dass $\mathfrak{D}$ selbst ein Regularitätsbereich ist. Die Beweise stützen sich im wesentlichen auf den erwähnten Satz von Behnke-Stein und den Cartan-Thullenschen Satz über die Regulärkonvexität [Math. Ann. **116**, 204–216 (1938); **106**, 617–647 (1932)]. [Vgl. auch die Note des gleichen Verfassers in Proc. Imp. Acad. Tokyo **17**, 7–10 (1941); diese Rev. **2**, 359.]

P. Thullen (Quito).

Peschl, E. Über einfach zusammenhängende Bereiche im Raume zweier komplexer Veränderlichen, die sich nicht auf echte Teilbereiche von sich analytisch abbilden lassen. Monatsh. Math. Phys. **51**, 63–84 (1943). [MF 14384]

Die Arbeit ist ein Beitrag zum Problem der "Starrheit" vierdimensionaler Bereiche des Raumes zweier komplexen Veränderlichen w, z gegenüber analytischen Abbildungen. Der Verfasser weist nach, dass es einfachzusammenhängende Regularitätsbereiche gibt (vom Zusammenhang der Hyperkugel), die ausser Automorphismen keine inneren analytischen Abbildungen zulassen; das heisst, es gibt keine analytische Abbildung, die einen solchen Bereich in einen von ihm verschiedenen, in seinem Innern gelegenen Bereich überführen würde. Als Beispiel gibt der Verfasser den Bereich $|wz(w+z-1)|<c$, mit $c>1/27$, an, der abgesehen von einer endlichen Gruppe trivialer Automorphismen keine inneren analytischen Abbildungen gestattet. Dabei werden analytische Abbildungen allgemeinster Art zugelassen, von denen nur vorausgesetzt wird, dass die Funktionaldeterminante nicht identisch verschwindet.

Wenn man von den Abbildungen ausserdem verlangt, dass sie topologisch sein sollen, so kann man einfachzusammenhängende Regularitätsbereiche angeben, die nicht nur keine analytischen topologischen Abbildungen auf echte Teilbereiche gestatten, sonder ausser der Identität auch keine Automorphismen zulassen. Die Beweise des Verfassers in einem und andern Falle stützen sich auf die Untersuchung der Carathéodoryschen Metrik und ihrer Entartungen.

P. Thullen (Quito).

Cartan, Henri. Idéaux de fonctions analytiques de n variables complexes. Ann. Sci. École Norm. Sup. (3) **61**, 149–197 (1944). [MF 14648]

Die vom Verfasser untersuchten Fragen bilden eine Erweiterung des bekannten Cousinschen Problems, zu einer in einem Bereiche D des Raumes R_{2n} vorgegebenen $(n-1)$-komplexdimensionalen Nullstellenfläche eine zugehörige, in D analytische Funktion zu konstruieren. Dieses und ähnliche Probleme lassen sich auf das folgende Grundproblem zurückführen. Jedem Punkt (x) eines gegebenen Bereiches D des Raumes R_{2n} sei eine in (x) analytische Funktion f_x so zugeordnet, dass die Gesamtheit dieser Funktionen der folgenden Verträglichkeitsbedingung genügt: zu jedem Punkt (a) aus D gebe es eine Umgebung V derart, dass f_a in V noch analytisch ist und die zu einem beliebigen (x) aus V gehörige Funktion f_x in (x) das gleiche Ideal wie f_a erzeugt (d.h. $f_x=\varphi f_a$ in (x), mit in (x) analytischem und dort nicht verschwindendem φ). Gibt es dann eine in ganz D analytische Funktion f, die in jedem (x) aus D das gleiche Ideal wie f_x erzeugt? Cartan dehnt diese Cousinsche Fragestellung auf beliebige Ideale und darüber hinaus auf "Moduln mehrdimensionaler Funktionen" aus. Eine solche Erweiterung gestattet, das Verhalten analytischer Flächen beliebiger komplexer Dimension im Grossen zu untersuchen. Aus der Fülle neuer Begriffe, Fragen und Ergebnisse seien einige herausgegriffen.

Ein System von q über einer Menge E analytischer Funktionen von n komplexen Veränderlichen kann als eine über E "analytische Funktion von q komplexen Dimensionen" definiert werden. In Analogie zum Begriffe der Ideale analytischer (eindimensionaler) Funktionen werden "Moduln" q-dimensionaler Funktionen eingeführt als Gesamtheiten $\mathfrak{M}$ von über einer gegebenen Punktmenge E analytischen q-dimensionalen Funktionen mit der Eigenschaft, dass die Summe zweier Elemente aus $\mathfrak{M}$ und ebenso das Produkt eines Elementes aus $\mathfrak{M}$ mit irgendeiner über E analytischen (eindimensionalen) Funktion wieder zu $\mathfrak{M}$ gehören. Das verallgemeinerte Cousinsche Problem ist jetzt wie folgt zu stellen. Zu jedem Punkt (x) eines Bereiches D sei ein in (x) analytischer Modul $\mathfrak{M}_x$ gegeben, deren Gesamtheit einer der Cousinschen analogen Verträglichkeitsbedingung genügen möge. Gibt es dann einen in ganz D analytischen Modul $\mathfrak{M}$, der in jedem (x) aus D den zugehörigen $\mathfrak{M}_x$ erzeugt? Zur Klärung dieser und anderer damit zusammenhängenden Fragen werden eine Reihe von Hilfsbegriffen eingeführt, wie der zu einem den Verträglichkeitsbedingungen genügenden System von Punktmodulen $\mathfrak{M}_x$ "assoziierte" Modul, der Begriff eines "vollkommenen" Moduls und der eines "schwach ableitbaren" und der eines "ableitbaren" Moduls schlechthin. Das verallgemeinerte Cousinsche Problem wird dann für Polyzylinder unter gewissen einschränkenden Bedingungen wie folgt gelöst. Sei Δ ein einfachzusammenhängender Polyzylinder, in dem ein den Verträglichkeitsbedingungen genügendes System von Moduln $\mathfrak{M}_x$ mit fol-

gender Eigenschaft gegeben sei: zu jedem Punkt von Δ gibt es eine abgeschlossene Umgebung V und über V ein dort $(n+1)$-mal schwach ableitbarer Modul, der in jedem Punkt (x) aus $\Delta \cap V$ den Modul $\mathfrak{M}_x$ erzeugt. Dann existiert ein in ganz Δ analytischer Modul $\mathfrak{P}$ (der zu den $\mathfrak{M}_x$ assoziierte), der in jedem (x) aus Δ den zugehörigen $\mathfrak{M}_x$ erzeugt. Der Modul $\mathfrak{P}$ ist vollkommen und, falls Δ abgeschlossen ist, eindeutig bestimmt und von endlicher Basis. Als Anwendung dieses Hauptsatzes wird untern anderm ein wichtiges Ergebnis von K. Oka abgeleitet und ein von A. Weil gestelltes Problem gelöst. Schliesslich werden die für abgeschlossene bzw. offene Polyzylinder erhaltenen Resultate auf Polyeder-Bereiche bzw. allgemeine schlichte Regularitätsbereiche übertragen (was für den Cousinschen Satz im engeren Sinne nicht möglich ist). Im besonderen folgt, dass jede in einem schlichten Regularitätsbereiche D gegebene $(n-1)$-komplexdimensionale analytische Fläche dort als gemeinsame Nullstellenfläche einer in D analytischen Funktionsfamilie dargestellt werden kann. *P. Thullen.*

Tornehave, Hans. On regular functions of several variables. The introduction of Jensen's function. Mat. Tidsskr. B. 1945, 151–155 (1945). (Danish) [MF 14269]

In the case of two complex variables $z_1=x_1+iy_1, z_2=x_2+iy_2$, let A be a two-dimensional domain in the "real" plane of (x_1, x_2) and let D be a four-dimensional neighborhood of A. Let $f(z_1, z_2)$ be an analytic function in D and let $z_2=h(z_1)=h_1(x_1, y_1)+ih_2(x_1, y_1)$ be a zero-manifold of $f(z_1, z_2)$. The equations $x_2=h_1(x_1, y_1)$, $y_2=h_2(x_1, y_1)$ can be solved by functions $y_1=Y_1(x_1, x_2)$, $y_2=Y_2(x_1, x_2)$. Substitute the latter in the common value of $\partial h_1/\partial y_1 \equiv -\partial h_2/\partial x_1$, and denote the result by $\Delta(x_1, x_2)$. The author shows that the equations

$$\partial\varphi/\partial x_1=-(\Delta/|\Delta|)Y_2, \quad \partial\varphi/\partial x_2=(\Delta/|\Delta|)Y_1$$

have as a solution in A a function $\varphi(x_1, x_2)$ which is convex in each variable separately and such that the integral

$$\int_A (\partial^2\varphi/\partial x_1^2+\partial^2\varphi/\partial x_2^2)dxdy$$

is the two-dimensional Euclidean area of the given zero-manifold. *S. Bochner* (Princeton, N. J.).

Poor, Vincent C. Complex functions possessing differentials. Amer. J. Math. 68, 147–160 (1946). [MF 15497]

The paper deals with complex functions $f(z, z')$ of two complex variables. Certain definitions of differentiability are considered. Unfortunately, the first theorem of the paper, dealing with the restricted Hamilton differential, is false. The second theorem deals with the Young definition of the differential. The status of this theorem is not clear to the reviewer because of obscurities in the definition and proof. *A. E. Taylor* (Los Angeles, Calif.).

Sugawara, Masao. Note on the fundamental domain of a general Fuchsian group. Proc. Imp. Acad. Tokyo 18, 12–13 (1942). [MF 14736]

Suppose that R is the set of all complex $n\times n$ matrices Z satisfying the inequality $N(Z)\leqq 1$ when $N(Z)=$l.u.b. $|Zx|$ for all n dimensional vectors x of length 1. Let G be a generalized Fuchsian group of substitutions

$$Z\to(U_1Z+U_2)(U_3Z+U_4)^{-1}$$

with

$$U'\begin{pmatrix}E_n & 0\\ 0 & -E_n\end{pmatrix}U=\begin{pmatrix}E_n & 0\\ 0 & -E_n\end{pmatrix}, \quad U=\begin{pmatrix}U_1 & U_2\\ U_3 & U_4\end{pmatrix},$$

U_i real. The author constructs a fundamental domain M_0 of G using the metric $\delta(A, B)=\{\mathrm{Sp}\,(A-B)'(A-B)\}^{\frac{1}{2}}$, $A, B\in R$. Let $M=R\cap[\cap_U M(U)]$, where $M(U)$ is the set of all $Z\in R$ with $\|U_3Z+U_4\|\geqq 1$ for $U_3\neq 0$. Then M_0 is the set of all $Z\in M$ with $\delta(Z, Z_1)\leqq\delta(Z, Z_i)$, Z_i conjugate to Z_1 with respect to G. [See also C. L. Siegel, Amer. J. Math. 65, 1–86 (1943), in particular, p. 27; these Rev. 4, 242.]

O. F. G. Schilling (Chicago, Ill.).

Takasu, Tsurusaburo. Einheitliche Theorie der Funktionen einer binären komplexen Veränderlichen. Proc. Imp. Acad. Tokyo 17, 349–357 (1941). [MF 14719]

A uniform theory of functions of a binary complex variable, $f(z)=f(x+jy)=u(x, y)+jv(x, y)$, with $j^2=\mu+\nu j$, $\nu^2+4\mu<0$, $=0$, >0, of elliptic, parabolic, or hyperbolic type, is sketched. Definitions and theorems are given relative to modulus, absolute value, convergence, and integration. An extensive bibliography is included.

E. F. Beckenbach (Los Angeles, Calif.).

Takasu, Tsurusaburo. Einheitliche Theorie der Funktionen einer binären komplexen Veränderlichen. II. Proc. Imp. Acad. Tokyo 17, 466–473 (1941). [MF 14728]

The work of the paper reviewed above is continued. Several proofs of analogues of the Cauchy integral formula, of the Morera theorem, and of the Beltrami-Pompeiu integral formula are given. *E. F. Beckenbach.*

Takasu, Tsurusaburo. Allgemeine Bahn der Theorie der Funktionen einer allgemeinen binären komplexen veränderlichen. Proc. Phys.-Math. Soc. Japan (3) 25, 209–220 (1943). [MF 15055]

The author studies functions of the binary complex variable $z=x+jy$, where $j^2=\mu+j\nu$. The numbers (μ, ν) are fixed real quantities and (x, y) are real independent variables. The binary variable z is elliptic, parabolic, or hyperbolic, according as $\nu^2+4\mu$ is negative, infinitesimal, or positive. Capelli has studied polygenic functions $w=F(z)$ of a binary variable. The author presents some theorems in the monogenic case, that is, when dw/dz exists and is unique in a region of the z-plane. The analogues of the Cauchy-Riemann equations, the Laplace equation, the Cauchy integral theorem, Morera's theorem and the Cauchy integral formula are discussed. Finally, the isotropic coordinates of Capelli are considered. The paper has numerous references to the literature on the subject. *J. DeCicco* (Chicago, Ill.).

Pyle, H. R., and Barker, Beverly M. A vector interpretation of the derivative circle. Amer. Math. Monthly 53, 79–82 (1946). [MF 15464]

The mean and phase derivatives of a polygenic function $F(z)$ are interpreted geometrically as vectors along gradients of certain contour lines. *E. F. Beckenbach.*

Theory of Series

Guinand, A. P. An asymptotic series for computing π. Math. Gaz. 29, 214–218 (1945). [MF 15218]

The formula established is

$$\tfrac{1}{4}\pi=\sum_{n=0}^{2N-1}\frac{(-1)^n}{2n+1}+\tfrac{1}{2}\sum_{n=0}^{k}\frac{E_{2n}}{(4N)^{2n+1}}+R_k,$$

where E_{2n} is an Euler number and

$$|R_k|<(4k+5)(2k+1)!/(2\pi N)^{2k+2}.$$

T. Fort (Athens, Ga.).

Pedersen, Peder. Continued fraction development for π. Mat. Tidsskr. B. 1945, 142–144 (1945). (Danish) [MF 14267]

The first 96 partial quotients in the regular continued fraction expansion of π are given, together with the 96th convergent. They are obtained from a 100 decimal value of π, and are in error by less than $1.6\cdot10^{-100}$. These values agree with a recent similar calculation of the first 100 quotients [Lehmer, Amer. Math. Monthly **45**, 227–233 (1938); **46**, 148–152 (1939)] with which the author was unfamiliar. There is no indication of the precise method used.

D. H. Lehmer (Aberdeen Proving Ground, Md.).

Puri, Amrit Sagar. An identity and some deductions. Math. Student **13**, 41–42 (1945). [MF 15639]

Simple remarks on sequences associated with the Fibonacci numbers. *I. Kaplansky* (Chicago, Ill.).

Singh, Daljit. On the series $\sum s^r$. Math. Student **13**, 59–60 (1945). [MF 15646]

The author gives an expression for $1^r+\cdots+n^r$ in terms of Stirling numbers. The result is known [C. Jordan, Calculus of Finite Differences, Budapest, 1939, p. 181].

I. Kaplansky (Chicago, Ill.).

Féraud, Lucien. Sur la distribution rectangulaire et les nombres de Bernoulli. C. R. Séances Soc. Phys. Hist. Nat. Genève **62**, 71–75 (1945). [MF 14561]

The relation between moments and seminvariants of the rectangular distribution gives rise to an identity on Bernoulli numbers which the author establishes directly.

I. Kaplansky (Chicago, Ill.).

Schuh, Fred. Investigation of the convergence or divergence of the series

$$\sum_{n=3}^{\infty}\log\left\{1+(-1)^n\Big/\prod_{j=0}^{p}(\log_j n)^{k_j}\right\}.$$

Mathematica, Zutphen. A. **13**, 21–24 (1944). (Dutch) [MF 15723]

Here $\log_0 n=n$, $\log_{j+1}n=\log\log_j n$. It is shown that the series converges if and only if the k_j are positive, not all equal to $\frac{1}{2}$, and the first one which differs from $\frac{1}{2}$ is larger than $\frac{1}{2}$. Other properties of the series are discussed.

R. P. Boas, Jr. (Providence, R. I.).

Schuh, Fred. Is the convergence criterion of Cauchy sharper than that of d'Alembert? Mathematica, Zutphen. A. **13**, 5–15 (1944). (Dutch) [MF 15722]

Comparison of the nth root test and the ratio test for convergence. *R. P. Boas, Jr.* (Providence, R. I.).

Hamming, R. W. Monotone series. Amer. J. Math. **68**, 133–136 (1946). [MF 15495]

Let $\sum c_n$ be a monotone convergent series and $\sum d_n$ a monotone divergent series. The object of the paper is to study the behavior of c_n/d_n. In particular, a function $A(n)$ is set up which is defined as the "average number" of times $c_n\geqq d_n$ for two particular series. Several theorems about $A(n)$ are proved. The most interesting result is that it is possible to have $\limsup A(n)=1$. *T. Fort.*

Agnew, Ralph Palmer. A simple sufficient condition that a method of summability be stronger than convergence. Bull. Amer. Math. Soc. **52**, 128–132 (1946). [MF 15452]

The author shows, by constructing an example, that there is at least one divergent sequence of zeros and ones which is summable by method A of matrix a_{nk}, if $\sum_{k=1}^{\infty}|a_{nk}|<\infty$, $n=1, 2, \cdots$, and $\lim_{n\to\infty}\max_{k=1,2,\cdots}|a_{nk}|=0$. From this follows the theorem that, if A is regular and satisfies the condition $\lim_{n,k\to\infty}a_{nk}=0$, then there are divergent sequences of zeros and ones which are summable A. *D. Moskovitz.*

Kakutani, Shizuo. Notes on divergent series and integrals. Proc. Imp. Acad. Tokyo **20**, 74–76 (1944). [MF 14875]

Theorem 1. Let $x(t)$ and $y(t)$ be two real nonnegative measurable functions defined over $0\leqq t\leqq1$. If $\int_E x(t)dt<\infty$ whenever E is a measurable subset of $0\leqq t\leqq1$ for which $\int_E y(t)dt<\infty$, then there exist a constant K and a real nonnegative function $z(t)$, integrable over $0\leqq t\leqq1$, such that $x(t)\leqq Ky(t)+z(t)$. Theorem 2. Let $a(1)+a(2)+\cdots$ and $b(1)+b(2)+\cdots$ be two series of real nonnegative numbers not greater than 1. If $\sum a(n_k)<\infty$ whenever $n_1, n_2, \cdots$ is an increasing sequence of integers for which $\sum b(n_k)<\infty$, then there exist a constant K and a convergent series $c(1)+c(2)+\cdots$ of real nonnegative numbers such that $a(n)\leqq Kb(n)+c(n)$. These two theorems are obtained by use of a third analogous theorem involving measures.

R. P. Agnew (Ithaca, N. Y.).

Selberg, Henrik L. Über die Darstellung willkürlicher Funktionen durch Charliersche Differenzreihen. Skand. Aktuarietidskr. **25**, 228–246 (1942). [MF 14139]

Let $\Theta(n)$ be a function of integers, zero for $n<0$. Let $v(n)$ satisfy (*) $\lim_{n\to\infty}n^k2^nv(n)=0$ for every positive k, and define

$$g(z)=\sum_{n=0}^{\infty}\Theta(n)z^n.$$

If $g(z)$ is analytic at $z=0$ and meromorphic and without zeros in $|z-1|\leqq1$, then for $n=0, 1, 2, \cdots$

$$(*^*)\qquad v(n)=\sum_{k=0}^{\infty}c_k\Delta^k\Theta(n),\quad c_k=\sum_{p=0}^{\infty}G_k(p)v(p),$$

where

$$G_k(z)=\frac{(-1)^n}{n!}\frac{d^n}{dz^n}\left(\frac{z^n}{g(z)}\right)_{z=1}$$

and $\Delta^k\Theta(n)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\Theta(n-j)$. If $v(n)\geqq0$ and $g(z)$ has no poles, the representation $(*^*)$ can hold only if (*) does. Under stronger hypotheses uniform convergence and uniqueness of $(*^*)$ can be proved. These results generalize theorems of E. Schmidt [Z. Angew. Math. Mech. **13**, 139–142 (1933)] and Uspensky [Ann. of Math. (2) **32**, 306–312 (1931)] for the special case $\Theta(n)=e^{-\lambda}\lambda^n/n!$. Applications are made to the theory of probability, where (for example) X' and X_Θ are independent integer-valued random variables, $X=X_\Theta+X'$, and $v(n)$ and $\Theta(n)$ are, respectively, the probabilities of $X=n$, $X_\Theta=n$. *R. P. Boas, Jr.*

Izumi, Shin-ichi, and Sunouchi, Gen-ichiro. A note on infinite series. Proc. Imp. Acad. Tokyo **18**, 532–534 (1942). [MF 14775]

The authors prove that, if p_n and q_n are the numbers of positive and negative terms in $s_n\equiv a_1+a_2+\cdots+a_n$, and if $s_n-(p_n-q_n)|a_n|\to0$ as $n\to\infty$, then $s_n\to0$ if either (i) $|a_n|\downarrow0$ and $1/(|a_n|^{-1}-|a_{n-1}|^{-1})=O(1)$ or (ii) $\sum_{n=1}^{\infty}(p_n-q_n)^{-2}<\infty$ and $(p_n-q_n)\sum_{m=n}^{\infty}(p_m-q_m)^{-2}\to0$. An example is given to show that the conclusion is false under the single condition $|a_n|\downarrow0$.

If $f(z)$ is an analytic function in the unit circle and $r_1, \cdots, r_n$ and $\rho_{n+1}, \cdots, \rho_{n+m}$ the absolute values of its zeros and poles, respectively, in $|z|<R<1$, the authors arrange r_i and ρ_j in increasing order, put $\log(1/r_i)=a_i$, $\log\rho_j=a_j$ and de-

duce from Jensen's theorem that

$$(2\pi)^{-1}\int_0^{2\pi}\log|f(Re^{i\phi})|\,d\phi=s_k-(p_k-q_k)|a_k|,$$

where $k=m+n$. The existence of

$$\lim_{R\to 1}(2\pi)^{-1}\int_0^{2\pi}\log|f(Re^{i\phi})|\,d\phi$$

then follows from that of $\lim_{n\to\infty}s_n$ when p_k, q_k or a_k satisfy the conditions of the theorem above. *H. R. Pitt.*

Shtsheglov, M. On some equalities. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 321–328 (1945). (Russian. English summary) [MF 15362]

Given a series $\sum a_n$ and a sequence of numbers $\lambda_n\uparrow\infty$, the author investigates conditions under which

$$(*)\qquad \liminf\sum a_n\sigma^{-\lambda_n\sigma}=\liminf\sum_{n\leq 1/\sigma}a_n,\qquad \sigma\to+0,$$

and a similar relation $(*^*)$ for the lim sup. The following theorem may serve as an example of the results achieved. If $\lambda_{n+1}/\lambda_n=O(1)$ and $a_n=o\{(\lambda_{n+1}-\lambda_n)/\lambda_n\}$, then both $(*)$ and $(*^*)$ hold [see also the author's paper in Rec. Math. [Mat. Sbornik] N.S. **14**(56), 109–132 (1944); these Rev. **6**, 210; Landau, Monatsh. Math. Phys. **18**, 8–28 (1907)].

A. Zygmund (Philadelphia, Pa.).

Kamber, Franz. Formules exprimant les valeurs des coefficients des séries de puissances inverses. Acta Math. **78**, 193–204 (1946). [MF 15602]

Let there be given $x^m=y^n(1+mT)$, where

$$T=c_1y+c_2y^2+c_3y^3+\cdots$$

and m is any complex number. The author obtains developments $y^n=x^n(1+s_n)$, where $s_n=b_{n,1}x+b_{n,2}x^2+b_{n,3}x^3+\cdots$. The purpose of the paper is to express the b's in terms of the c's. To do this he begins with Lagrange's classical formula for the inverse in $y=y_0+xf(y)$ and obtains determinant formulas for the desired coefficients. These are applied to an example. A special discussion is given for the case that $m=n=1$. Finally, formulas in determinant form are obtained for the successive derivatives of x with respect to y in terms of the derivatives of y with respect to x.

T. Fort (Athens, Ga.).

Fourier Series and Generalizations, Integral Transforms

Hsiang, Fu Cheng. The summability $(C,1-\epsilon)$ of Fourier series. Duke Math. J. **13**, 43–50 (1946). [MF 15872]

Let $(*)$ $f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ and let $\varphi(u)=f(x+u)+f(x-u)-2f(x)$. The convergence of the integral $\int_0^t u^{-1}\varphi(u)du$ implies summability $(C,1+\eta)$ of $(*)$ for all $\eta>0$, but is not sufficient for summability $(C,1)$ [B. N. Prasad, Math. Z. **40**, 496–502 (1935)]. The convergence of $\int_0^t u^{-1-\epsilon}\varphi(u)du$ implies summability $(C,1)$, but does not imply summability $(C,1-\epsilon)$ if $\epsilon>\eta/(1+\eta)$.

P. Civin (Buffalo, N. Y.).

Zaanen, A. C. On the absolute convergence of Fourier series. Nederl. Akad. Wetensch., Proc. **48**, 211–215 = Indagationes Math. **7**, 16–20 (1945). [MF 15792]

Let a_n, b_n be the Fourier coefficients of a function $f(x)$. Szász proved that, if $f\in\mathrm{Lip}\,\alpha$, then $\sum\{|a_n|^\beta+|b_n|^\beta\}$ converges for $\beta>2/(2\alpha+1)$ [see, for example, the reviewer's "Trigonometrical Series," Warsaw-Lwów, 1935, p. 137]. The author generalizes the result by introducing logarithmic factors into the modulus of continuity of f. The generalization is a special case of another theorem of Szász [Trans. Amer. Math. Soc. **42**, 366–395 (1937), in particular, p. 376, theorem 3.1; see also Hille and Tamarkin, Math. Ann. **108**, 525–577 (1933)]. An example is given to illustrate the generalization. *A. Zygmund* (Philadelphia, Pa.).

Bödewadt, U. T. Die Fourierentwicklung des Sinus, Cosinus und der Umkehrung einer Fourierreihe. Math. Z. **47**, 655–662 (1942). [MF 15908]

A result of P. Lévy [C. R. Acad. Sci. Paris **196**, 463–464 (1933); Compositio Math. **1**, 1–14 (1933)] states that, if $y=\varphi(t)$ has an absolutely convergent Fourier series, if R is an open set in the y-plane which contains the domain of the function $\varphi(t)$, and if $f(y)$ is a function analytic in R, then $g(t)=f(\varphi(t))$ has an absolutely convergent Fourier series. The present paper gives explicit formulas for the coefficients of the series for $g(t)$ in terms of Bessel functions of integral order in the case when $f(y)$ is either $\sin y$ or $\cos y$. These expansions are used to obtain, under suitable hypotheses, the explicit expansion for the Fourier series of the inverse of a given function. *P. Civin* (Buffalo, N. Y.).

Grünwald, Géza. Eine Bemerkung zu meiner Arbeit "Über die Summabilität der Fourierschen Reihe." Acta Litt. Sci. Szeged **10**, 105–108 (1941). [MF 15837]

Let $f(x,y)$ be an L integrable function of period 2π with respect to each variable and let $s_{mn}(x,y)$ denote the (rectangular) partial sums of the Fourier series of $f(x,y)$. It is well known that the sequence of the (square) partial sums $s_{nn}(x,y)$ is almost everywhere summable to sum $f(x,y)$ by various methods, in particular by the method $(C,1)$ [see L. Fejér, Proc. Cambridge Philos. Soc. **34**, 503–509 (1938); J. Marcinkiewicz, Ann. Scuola Norm. Super. Pisa (2) **8**, 149–160 (1939); G. Grünwald, Proc. Cambridge Philos. Soc. **35**, 343–350 (1939); J. G. Herriot, Trans. Amer. Math. Soc. **53**, 72–94 (1942); these Rev. **1**, 51; **4**, 38]. The author now shows that

$$\{s_{0,n}(x,y)+s_{1,n-1}(x,y)+\cdots+s_{n,0}(x,y)\}/(n+1)\to f(x,y)$$

at almost every point. *A. Zygmund* (Philadelphia, Pa.).

Alexits, Georges. Sur la sommation forte des séries orthogonales. Comment. Math. Helv. **18**, 122–128 (1946).

Let $\sigma_n^\alpha(x)$ be the (C,α) means of an orthonormal series $\sum a_k\varphi_k(x)$. The series is said to be strongly summable $(C,1+\alpha)$ at the point x, to sum $f(x)$, if

$$\sum_{\nu=0}^{n}|\sigma_\nu^\alpha(x)-f(x)|=o(n).$$

It is shown that, if $\sum a_k^2k^{-2\alpha}$ converges, $-1<\alpha\leqq 0$, the series $\sum a_k\varphi_k$ is strongly summable $(C,1+\alpha)$ at almost every point where it is summable by Abel's method.

A. Zygmund (Philadelphia, Pa.).

Pollard, Harry. The mean convergence of orthogonal series of polynomials. Proc. Nat. Acad. Sci. U. S. A. **32**, 8–10 (1946). [MF 15311]

The following results are announced. (i) Let

$$p(x)=(1-x^2)^{\lambda-\frac{1}{2}},$$

$-1<x<1$, $\lambda\geqq 0$ and let $P_n(x)$ be the (ultraspherical) orthonormal polynomials associated with the weight function

$p(x)$. Let $f(x)\sim\sum a_nP_n(x)$, so that $a_n=\int_{-1}^{1}f(x)P_n(x)p(x)dx$, and let $s_n(x)$ denote the partial sums of the latter series. Suppose that $\int_{-1}^{1}|f|^r p dx<\infty$. Then

$$\int_{-1}^{1}|f-s_n|^r p dx\to 0, \qquad n\to\infty,$$

for $2-(\lambda+1)^{-1}<r<2+\lambda^{-1}$, but not necessarily for $r<2-(\lambda+1)^{-1}$ or for $r>2+\lambda^{-1}$; for $\lambda=0$, we get a well-known result of M. Riesz [Math. Z. **27**, 218–277 (1927)]. (ii) Let $q(x)=(1-x)^\alpha(1+x)^\beta$, $-1<x<1$, $\alpha\geqq 0$, $\beta\geqq 0$, and let $Q_n(x)$ be the (Jacobi) orthonormal polynomials with respect to the weight function $q(x)$, so that $q^{\frac{1}{2}}(x)Q_n(x)$ form an orthonormal system in the ordinary sense. Let $t_n(x)$ be the partial sums of the Fourier series $\sum b_nq^{\frac{1}{2}}(x)Q_n(x)$ of any function $f(x)$, $-1<x<1$ (thus $b_n=\int_{-1}^{1}fq^{\frac{1}{2}}Q_n dx$). Suppose that $\int_{-1}^{1}|f|^r dx<\infty$. Then

$$\int_{-1}^{1}|f-s_n|^r dx\to 0, \qquad n\to\infty,$$

for $4/3<r<4$, but not if $r<4/3$ or if $r>4$.

A. Zygmund (Philadelphia, Pa.).

Bourgin, D. G. On certain sequences of functions. Proc. Nat. Acad. Sci. U. S. A. **32**, 1–5 (1946). [MF 15309]

A number of results are announced which extend results of the author and Mendel [Trans. Amer. Math. Soc. **57**, 332–363 (1945); these Rev. **6**, 266] on conditions for a set of functions $\{f(nx)\}$ to be orthonormal. Let $f(x)$ be odd and let a_k be its Fourier sine coefficients. Many of the previous results referring to the case where $\sum|a_n|<\infty$ can be extended to cases where $\sum a_n n^{-it}$ converges uniformly or $\sum|a_n|^2<\infty$. Further results are stated, and connections with the theory of general transforms (in the sense of Watson) are given. Some new completeness theorems are also stated. For example, let $f(x)$ have the form

$$\sin x+\sum_1^M b_k\sin kx.$$

Let $B_{i,j}=\sum_{k=1}^{m}b_{i,k}b_{j,k}$. If at least two b_k's are different from 0, the set $\{f(nx)\}$ is complete if

$$\sum_{(i,j)=1}B_{i,j}\cos t\log(i/j)\leqq 1, \qquad -\infty<t<\infty.$$

R. P. Boas, Jr. (Providence, R. I.).

Fuchs, W. H. J. On the closure of $\{e^{-t}t^{a_r}\}$. Proc. Cambridge Philos. Soc. **42**, 91–105 (1946). [MF 15665]

The author proves that the set of the title, with $a_r>0$, $a_{r+1}-a_r>c>0$, is closed in $L^2(0,\infty)$ if and only if $\int_{a_1}^{\infty}\psi(r)r^{-2}dr=\infty$, where $\psi(r)=\exp\{2\sum_{a_n>r}a_r^{-1}\}$, $r>a_1$. This extends his earlier result [same Proc. **40**, 188–196 (1944); these Rev. **6**, 46, 334]. The proof depends on the following lemma. If $g(z)$ is analytic in $\Re(z)\geqq 0$, $|g(re^{i\theta})|\leqq\{x/\psi(r)\}^x$, and $\int^{\infty}\psi(r)r^{-2}dr=\infty$, then $g(z)\equiv 0$. *R. P. Boas, Jr.*

***Schwartz, Laurent. Étude des sommes d'exponentielles réelles.** Actualités Sci. Ind., no. 959. Hermann et Cie., Paris, 1943. 89 pp.

According to the theorem of Müntz and Szász the set $\{t^{\mu_n}\}$, under some mild restrictions on the μ_n, is complete in $L^p(0,1)$ if and only if $\sum 1/\mu_n=\infty$. The present study gives a proof of this theorem and then proceeds to an investigation of the case in which $\sum 1/\mu_n$ converges. Under this hypothesis the L^p closure of the linear combinations of t^{μ_n} is not the whole space; let it be denoted by A^p. Theorems of the following character are proved. If $\mu_n+1/p>0$, $\lim_{n\to\infty}\mu_n=\infty$, the index of condensation of $\{\mu_n\}$ is zero and $\sum 1/\mu_n$ converges, then every element $f(x)$ of A^p is analytic on $(0,1)$ and can be continued into the region $|z|<1$ of the logarithmic Riemann surface; moreover, every such $f(z)$ has an expansion $\sum c_nz^{\mu_n}$ which converges to it "normally" in every region $|z|\leqq 1-\epsilon$. A series converges normally if the series of absolute values converges uniformly. [In case the μ_n are integers a similar theorem was proved independently at about the same time by Clarkson and Erdös, Duke Math. J. **10**, 5–11 (1943); these Rev. **4**, 196].

A chapter is devoted to inequalities of the Markoff-Bernstein type for polynomials in t^{λ_n}. Asymptotic bounds are given for the coefficients of such polynomials with given L^p norm; the bounds are exact when $p=2$. [For first derivatives and ordinary polynomials, cf. Hille, Szegö and Tamarkin, Duke Math. J. **3**, 729–739 (1937)].

[On p. 24 the author lists the literature on the Müntz-Szász theorem, attributing different proofs to Kaczmarz-Steinhaus and Szász. These proofs are actually the same.]

H. Pollard (New Haven, Conn.).

de Bruijn, N. G. On the absolute convergence of Dirichlet series. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 23–28 (1943). (Dutch. German, English and French summaries) [MF 15773]

Let $f(s)=\sum_1^{\infty}a_ne^{-\lambda_n s}$, $\lambda_n\uparrow\infty$, $1/(\lambda_{n+1}-\lambda_n)=O(e^{\lambda_n(1+\delta)})$, $\sigma\geqq\sigma_0$, where $s=\sigma+it$. Let $f(s)=O(|t|^k)$, $|t|\to\infty$, and

$$\int_{-T}^{T}|f(\sigma_0+it)|^2dt=O(T^{2k+1}).$$

Then the Dirichlet series is absolutely convergent for $\sigma>\sigma_0+(k+\frac{1}{2})l$. This improves a result of Grandjot [Math. Z. **26**, 593–618 (1927)] and is the best possible result of its kind. *R. P. Boas, Jr.* (Providence, R. I.).

***Ríos, Sixto. La Prolongación Analítica de la Integral de Dirichlet-Stieltjes. [The Analytic Continuation of the Dirichlet-Stieltjes Integral].** Consejo Superior de Investigaciones Cientificas, Madrid, 1944. 93 pp. (Spanish)

This appears to be a revision of the author's lectures [Revista Acad. Ci. Madrid **37** (1943); these Rev. **7**, 61]. The author now uses the "generalized" integral $\int_0^{\infty}e^{-\lambda(t)s}d\alpha(t)$, $\lambda(t)$ nondecreasing, instead of $\int_0^{\infty}e^{-ts}d\alpha(t)$. Some additional topics are presented. These include a discussion of the class of all Laplace-Stieltjes transforms as a complete metric space, the distance between two elements being e^{σ} if the abscissa of convergence of their difference is C, elements at zero distance being identified. The author also discusses the approximation of analytic functions by exponential polynomials. For example, if D is a simply-connected region having at least two boundary points and not containing ∞, any $f(z)$ analytic in D can be approximated by a sequence of exponential polynomials with rational exponents, converging uniformly in any closed subregion. Furthermore, a Dirichlet series can be found for which a sequence of partial sums converges uniformly in every closed subregion to a function for which D is the complete domain of regularity. *R. P. Boas, Jr.*

Korevaar, J. Some entire functions represented in a half-place by Laplace integrals. Mathematica, Zutphen. B. **12**, 107–114 (1944). (Dutch) [MF 15719]

Let $g(s)=\int_1^{\infty}G(x)x^{-s-1}dx$. (I) If $G(x)$ is integrable and periodic, not zero almost everywhere, with mean value γ over a period, then the abscissa of convergence of the integral is either 0 or 1, but the analytic continuation of $g(s)-\gamma s^{-1}$ is entire. (II) With the same hypotheses on $G(x)$, the ana-

lytic continuation of

$$\int_1^\infty x^\lambda(G(x)-\gamma)e^{-x^\alpha}dx$$

is entire, where λ is real and $0<\alpha<1$. (III) If $G(x)$ is the Fourier transform of a function which is zero in the neighborhood of 0 and satisfies a suitable integrability condition, then the analytic continuation of $g(s)$ is entire.

R. P. Boas, Jr. (Providence, R. I.).

Gupta, H. C. Some new kernels for the derivation of self-reciprocal functions. Proc. Indian Acad. Sci., Sect. A. 22, 228–233 (1945). [MF 14449]

If $f(x)$ is R_ν (self-reciprocal in the Hankel transform of order ν) there exist kernels $P(x)$ such that

$$g(x)=\int_0^\infty P(xy)f(y)dy$$

is R_μ. The author calls such kernels $C(\nu, \mu)$ and uses known integrals to obtain kernels from which new R_ν functions may be derived. Two of his simplest kernels are: $I_n(\frac{1}{2}x)K_n(\frac{1}{2}x)$, $\Re(n)>-\frac{1}{4}$, which is a $C(2n, \pm\frac{1}{2})$; $x^\frac{1}{2}J_\nu(2^{-\frac{1}{2}}x)K_\nu(2^{-\frac{1}{2}}x)$, $\Re(\nu)>-\frac{1}{2}$, which is a $C(0, 2\nu)$. *M. C. Gray.*

Polynomials, Polynomial Approximations

Rosenbloom, P. C. An elementary constructive proof of the fundamental theorem of algebra. Amer. Math. Monthly 52, 562–570 (1945). [MF 14487]

Many proofs of the fundamental theorem of algebra involve notions which are only in part algebraic. Given a polynomial $P(x)$, the author constructs by rational methods a sequence of points which satisfy the Cauchy convergence criterion and such that the polynomial tends to zero. The only limiting process which is used is in the definition of the real number system. The author replaces integration by summation around a square, roughly paralleling one of the proofs of the fundamental theorem of algebra which are given in function theory. *A. C. Schaeffer.*

Marden, Morris. A note on the zeros of the sections of a partial fraction. Bull. Amer. Math. Soc. 51, 935–940 (1945). [MF 14464]

The zeros of $\sum_{k=1}^3 m_k/(z-z_k)$, where z_1, z_2, and z_3 are distinct noncollinear points, and m_1, m_2, and m_3 are real numbers different from zero, lie at the foci of the conic which touches the sides of the triangle (z_1, z_2, z_3) at the points which divide them in the ratios $m_1:m_2$, $m_2:m_3$ and $m_3:m_1$, respectively. Generalizations to relations between the zeros of $F_1(z)$, $F_2(z)$, and $F_1(z)+F_2(z)$, where

$$F_\nu(z)=\sum_{k=1}^{n_\nu} m_{k,\nu}/(z-z_{k,\nu}),$$

$\nu=1, 2$, are also given, with geometrical interpretations.

P. C. Rosenbloom (Providence, R. I.).

Frank, Evelyn. On the zeros of polynomials with complex coefficients. Bull. Amer. Math. Soc. 52, 144–157 (1946). [MF 15455]

Following the methods used for real polynomials by H. S. Wall [Amer. Math. Monthly 52, 308–322 (1945); these Rev. 7, 62], a continued-fraction criterion is developed for a polynomial $P(z)=z^n+c_1z^{n-1}+\cdots$ with complex coefficients $c_j=p_j+iq_j$ to have exactly $m\leqq n$ zeros in the half-plane $\Re(z)<0$. The polynomial $P(z)$ has this property if, on its division into $Q(z)=\frac{1}{2}\{P(z)-(-1)^nP(-z)\}$, a continued fraction

$$\frac{Q(z)}{P(z)}=\cfrac{a_0}{z+a_0+\cfrac{b_1}{z+b_2+\cfrac{a_2}{z+b_3+\cdots+\cfrac{a_{n-1}}{z+b_n}}}}$$

is obtained with m positive a_j, $n-m$ negative a_j and n pure imaginary b_j. An equivalent requirement is that m variations of sign occur in the sequence of determinants $D_1=p_1$,

$$D_k=\begin{vmatrix} p_1, p_2, p_3, \cdots, p_{2k-1}, -q_2, -q_4, \cdots, -q_{2k-2} \\ 1, p_2, p_4, \cdots, p_{2k-2}, -q_1, -q_3, \cdots, -q_{2k-3} \\ \cdots \quad\quad \cdots \\ 0, 0, 0, \cdots, p_k, 0, \cdots, -q_{k-1} \\ 0, q_2, q_4, \cdots, q_{2k-2}, p_1, p_3, \cdots, p_{2k-2} \\ 0, q_1, q_3, \cdots, q_{2k-3}, 1, p_2, \cdots, p_{2k-4} \\ \cdots \quad\quad \cdots \\ 0, \cdots, q_k, 0, \cdots, p_{k-1} \end{vmatrix},$$

$k=2, 3, \cdots, n$. The latter result coincides with that due to H. Bilharz [Z. Angew. Math. Mech. 24, 77–82 (1944); these Rev. 7, 62] and reduces to the familiar Hurwitz criterion when all the $q_j=0$. *M. Marden* (Milwaukee, Wis.).

v. Sz. Nagy, Gyula. Verallgemeinerung eines Satzes von Jentzsch. Monatsh. Math. Phys. 51, 59–62 (1943). [MF 14383]

Let $F_1(z)=f(z)-g_1(z)$ and $F_2(z)=f(z)-g_2(z)$, where $f(z)$, $g_1(z)$ and $g_2(z)$ are polynomials of degrees n, m_1 and m_2, respectively, and $m_1\leqq m\leqq n-1$ and $m_2\leqq m$. Let $F(z)$ be defined as $(k_1+k_2)F(z)=k_1F_1(z)+k_2F_2(z)$. As proved by M. Marden [Trans. Amer. Math. Soc. 32, 658–668 (1930)], if all the zeros of $F_1(z)$ and $F_2(z)$ lie in the same convex region B and if $0<|\arg(-k_1/k_2)|=\varphi\leqq\pi$, then all the zeros of $F(z)$ lie in the region comprised of all points from which B subtends an angle of at least $\varphi/(m+1)$. In the present paper, B is taken as a circle of radius R and the concentric circle of radius $R_1=R(1+p)/(1-p)$ is shown to contain all the zeros of $F(z)$, provided $p^{m+1}=\min(k_1/k_2, k_2/k_1)\neq 1$. The proof is based upon two lemmas: if $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ has all its zeros in a circular region C, not containing the point z_0, then each equation $Z=z_0-nf(z_0)/f'(z_0)$ [Laguerre, Oeuvres, vol. 1, Paris, 1898, pp. 56–63], $f(z_0)=(z_0-Z)^n$ [Walsh, Trans. Amer. Math. Soc. 24, 163–180 (1922)] has as solution a point Z in C. *M. Marden.*

***Parodi, Maurice. Application des polynomes électrosphériques à l'étude des systèmes oscillants à un grand nombre de degrés de liberté.** Mémor. Sci. Phys., no. 47. Gauthier-Villars, Paris, 1944. 82 pp.

This booklet deals with the theory and applications of a class of polynomials analogous to the Chebyshev trigonometric polynomials. They are defined by

$$y_n(x)=\sum(-1)^rC(n-r, r)x^{n-2r}.$$

They satisfy the differential equation

$$(x^2-4)y_n''+3xy_n'=n(n+2)y_n$$

and the recurrence relation $y_n-xy_{n-1}+y_{n-2}=0$. They are called electrospherical polynomials by A. Guillet and M. Aubert, who applied them to electrostatic problems with central symmetry in another booklet in the same series [Propriétés électrostatiques des systèmes sphériques, Mémor. Sci. Phys., no. 38, Gauthier-Villars, Paris, 1938].

The author first develops the theory of the Lucas functions U_n and V_n, of which these polynomials are a special case [E. Lucas, Théorie des Nombres, Paris, 1891]. He emphasizes their representation by determinants since they occur that way in his applications. These applications, which are too numerous to be listed here, are largely to

problems of finding characteristic frequencies of oscillatory systems with many degrees of freedom, composed of identical iterated elements. They include, in electrical network theory, the determination of the critical frequencies of filters of various designs and, in chemistry, the vibration frequencies of straight-chain hydrocarbons and of some ring compounds. Other applications are to vacuum tube circuits and to the propagation of X-rays in crystals.

O. Frink (State College, Pa.).

Arpiarian, Noubar. Polynomes trigonométriques orthogonaux relatifs à une ellipse de foyers $(-1, +1)$. C. R. Acad. Sci. Paris **219**, 668–669 (1944). [MF 15301]

The following results are stated. Let $f(x)$ be any non-decreasing convex function, E any ellipse with foci at ± 1 and D its interior. Among all polynomials $P_n(z)=z^n+\cdots$, the Chebyshev polynomials $T_n(z)=2^{1-n}\cos(n \arccos z)$ realize the minima of the integrals

$$\int_E f(|P_n(z)|)\,|1-z^2|^{-\frac{1}{2}}ds, \quad \iint_D f(|P_n(z)|)\,|1-z^2|^{-1}dS.$$

They are orthogonal on E with respect to the weight-function $|1-z^2|^{-\frac{1}{2}}$ and in D with respect to $|1-z^2|^{-1}$. Weighted orthogonality is also stated on E and in D for the polynomials $2^{-n}\sin\{(n+1)\arccos z\}/\sin(\arccos z)$ with respect to $|1-z^2|^{\frac{1}{2}}$ on E and $|1-z^2|$ in D;

$$2^{-n}\sin\{(n+\tfrac{1}{2})\arccos z\}/\sin(2^{-1}\arccos z)$$

with respect to $|(1-z)/(1+z)|^{\frac{1}{2}}$ on E and $|(1-z)/(1+z)|$ in D; $2^{-n}\cos\{(n+\frac{1}{2})\arccos z\}/\cos(2^{-1}\arccos z)$ with respect to $|(1+z)/(1-z)|^{\frac{1}{2}}$ on E and to $|(1+z)/(1-z)|$ in D. The integrals of weighted squares of all these polynomials on E and in D are computed. *E. Kogbetliantz.*

Arpiarian, Noubar. Sur la théorie des fonctions orthogonales de variable complexe. C. R. Acad. Sci. Paris **219**, 387–389 (1944). [MF 15271]

Polynomials $P_n(z)$ and $Q_n(z)$, orthogonal, respectively, on an arbitrary closed rectifiable curve C and on the interior D of the same curve C, are known to exist and are uniquely determined. The author has established that in general there is no connection between them and found particular examples of curves C such that the corresponding $P_n(z)$ are not orthogonal in D and, respectively, the corresponding $Q_n(z)$ are not orthogonal on C.

Introducing a positive continuous weight-function $w(z)$ and considering weighted orthogonality, the author formulates a theorem, the first part of which [existence of uniquely determined $P_n(z)$ and $Q_n(z)$] does not seem to be new [see G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., vol. 23, New York, 1939, p. 357; these Rev. **1**, 14.] Developments of an analytic function $f(z)$ regular in D and continuous on C in series of polynomials $P_n(z)$ and $Q_n(z)$ are considered and their uniform convergence with the sum $f(z)$ in every closed domain interior to C, if $w(z)\geqq k>0$, is stated. *E. Kogbetliantz* (New York, N. Y.).

Special Functions

Timman, R. A generalization of Stirling's formula for the Γ-function. Mathematica, Zutphen. B. **11**, 102–108 (1943). (Dutch) [MF 15727]

Direct proof of an expansion for $\log\Gamma(z+a)$ [Whittaker and Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, 1927, p. 276].

Mursi, Zaki. Note on two asymptotic expressions for the associated Legendre function. Proc. Math. Phys. Soc. Egypt **1**, no. 4, 17–20 (1940). [MF 14108]

The author gives new proofs of the asymptotic formulas for $P_n^m(x)$ (i) when n is large, m and n are integral and $|x|<1$, (ii) for large real n, m real, $x>1$. [See also E. W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Cambridge University Press, 1931, pp. 105, 306.]

H. Pollard (New Haven, Conn.).

Shabde, N. G. On a definite integral involving Legendre functions. Proc. Benares Math. Soc. (N.S.) **5**, 31–32 (1943). [MF 14439]

The integral in question is

$$\int_{-1}^{1} P_m(x)P_n(x)P_p(x)P_q(x)dx = \sum_{s=0}^{n} 2\frac{2m+2n-4s+1}{2m+2n-2s+1} \times \frac{A_{m-s}A_sA_{n-s}A_{\frac{1}{2}(p+q-m-n)+s}A_{\frac{1}{2}(p-q+m+n)-s}A_{\frac{1}{2}(q-p+m+n)-s}}{(p+q+m+n-2s+1)A_{m+n-s}A_{\frac{1}{2}(p+q+m+n)-s}},$$

where $p-q\leqq m\pm n\leqq p+q$, $p+q+m+n$ is an even integer, and $m!A_m=1\cdot 3\cdot 5\cdot\cdots\cdot(2m-1)$. *A. Erdélyi.*

Gupta, H. C. Operational calculus and the evaluation of a certain class of definite integrals. Proc. Benares Math. Soc. (N.S.) **5**, 1–16 (1943). [MF 14437]

A number of definite integrals involving the hypergeometric functions are evaluated by means of the convolution theorem for Laplace integrals. For example, it is proved that for $\Re\nu>-\frac{1}{2}$

$$\int_0^{\pi/2} J_{2\nu}(2^{\frac{1}{2}}\cos\theta)\cos(\sin\theta)d\theta=\tfrac{1}{2}\pi J_\nu(2)J_\nu(1).$$

H. Pollard (New Haven, Conn.).

Wavre, Rolin. Sur l'équation de Mathieu. C. R. Séances Soc. Phys. Hist. Nat. Genève **62**, 54–55 (1945). [MF 14557]

By the introduction of a new independent variable which is an exponential function of the original one, Mathieu's equation is transformed into a second order differential equation with algebraic coefficients. By applying a power series solution to this equation, infinite determinants of von Koch's type are obtained. These are simplified and their rapid convergence is established. [Previous similar solutions: H. Bremekamp, Nieuw Arch. Wiskunde (2) **15**, 292–301 (1927).] *M. J. O. Strutt* (Eindhoven).

Bickley, W. G., and McLachlan, N. W. Mathieu functions of integral order and their tabulation. Mathematical Tables and Other Aids to Computation **2**, 1–11 (1946). [MF 14669]

The aims of the article are (a) to define a set of Mathieu functions appropriate to the solution of potential, wave and analogous problems, (b) to make suggestions as to which should be tabulated, (c) to indicate which formulas are most convenient for computation. In connection with (a) the usual equations and definitions of periodic Mathieu functions of the first kind and of integral order are recapitulated, showing the interconnections of equations associated with different signs of the parameters. The corresponding Mathieu functions of the second kind are mentioned briefly. Associated Mathieu functions of the first and second kind and of integral order are given in terms of their Bessel and

Neumann series expansions, omitting functions of the third kind (asymptotically similar to Hankel functions), which are of considerable importance in wave propagation problems. It is suggested that at first low-accuracy tables would be desirable for the functions mentioned and numerical illustrations of computation are given. [It seems doubtful if general agreement with the authors' notations will be forthcoming.] *M. J. O. Strutt* (Eindhoven).

Differential Equations

***Valiron, Georges. Équations Fonctionnelles. Applications.** Masson et Cie, Paris, 1945. ii+605 pp.

[This is part II of the author's Cours d'Analyse Mathématique. For part I, Théorie des Fonctions, see these Rev. 7, 283.] Ordinary differential equations, partial differential equations, and algebraic equations of two variables are treated in this volume. The title arises from the fact that each of these defines functions. The applications are to geometry.

In the first chapter analytic functions $u=f(z)$ defined by irreducible polynomials $a_n(z)u^n+\cdots+a_0(z)=0$, where $a_i(z)$ are polynomials, are treated, together with the associated Riemann surfaces. This forms a background for the following chapter on Abelian integrals. Chapter three is devoted to analytic functions of several variables and Cauchy's method of dominant functions. In preparation for the study of singular solutions of differential equations the next section of the book treats the theory of contact of curves and surfaces and envelopes. Chapters five through ten contain an excellent treatment of ordinary differential equations in both the real and complex domain. An introduction to the calculus of variations is given in chapter eleven. The next three chapters are given over to the theory of curves and surfaces, including the Frenet-Serret formulas, theorems of Meusnier, Bonnet, and Gauss. By way of completing the theory of differential and integral calculus, the last two chapters give the classical theory of first and second order partial differential equations. *F. G. Dressel.*

Henderson, Archibald. Differential equations with quadrilateral envelope—cuspidal and nodal loci. Nat. Math. Mag. **20**, 51–68 (1945). [MF 15410]

A study of particular differential equations of the type $F(x,y)p^2+G(x,y)p+H(x,y)=0$, $p=dy/dx$, with F, G, H polynomials in x and y so chosen that G^2-4FH breaks up into first degree factors. Each differential equation is reduced by a substitution to Clairaut's form $P^2-Pu+v=0$, $P=dv/du$. *P. Franklin* (Cambridge, Mass.).

Matsumoto, Toshizô. A note on Fowler's differential equation. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **24**, 79–81 (1944). [MF 15556]

The equation in question is

$$\frac{d}{d\xi}\left\{\xi^2\frac{d\theta}{d\xi}\right\}+\xi^\lambda\theta^n=0.$$

The theorem that a solution with $\theta(0+)=C>0$ has a zero in $0<\xi<\infty$ if and only if $2\lambda-n+1>0$ is ascribed to G. Sansone [Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. 1, 163–176 (1940)]. The author transforms the equation into the form $v''-v+4e^{(2\lambda-n+1)u}v^n=0$ and derives an identity used by Sansone. *R. P. Boas, Jr.*

Waidelich, D. L. The steady-state operational calculus. Proc. I. R. E. **34**, 78P–83P (1946).

This paper adapts the unilateral Laplace transform to the solution of ordinary linear differential equations with constant coefficients whose inhomogeneous term is a periodic function. Examples from electric circuit theory are discussed. *A. E. Heins* (Lafayette, Ind.).

Kamke, E. Über Sturms Vergleichssätze für homogene lineare Differentialgleichungen zweiter Ordnung und Systeme von zwei Differentialgleichungen erster Ordnung. Math. Z. **47**, 788–795 (1942). [MF 15915]

The paper appeared in English in Amer. Math. Monthly **46**, 417–421 (1939); these Rev. **1**, 54. Some additional remarks have been added.

Verstraete, Roland. Generalization of a theorem of E. Lindelöf. Wis- en Natuurk. Tijdschr. **12**, 87–90 (1945). (Dutch) [MF 15589]

The author obtains an interval in which the equation $dy/dx=f(x,y)$ has a unique solution going through $(0, y_0)$ under the hypotheses that $f(x,y)$ is continuous in $0<x\leqq a$, $|y-y_0|\leqq b$, $|f(x,y_1)-f(x,y_2)|\leqq kx^{-m}|y_1-y_2|$, $|f(x,y_0)|\leqq Mx^{-m}$, $0\leqq m<1$. Lindelöf considered the case $m=0$. *R. P. Boas, Jr.* (Providence, R. I.).

Dieudonné, Jean. Sur la convergence des approximations successives. Bull. Sci. Math. (2) **69**, 62–72 (1945). [MF 14429]

The author considers generalizations of the usual Lipschitz condition for the existence of a unique solution for a system of ordinary differential equations. His criterion is slightly more restrictive than that of Kamke [Differentialgleichungen Reeler Funktionen, Leipzig, 1930, p. 139] but he proves not only uniqueness but also that the Picard successive approximations converge to the solution. His condition includes as special cases those of Osgood [Monatsh. Math. Phys. **9**, 331–345 (1898)], Nagumo [Jap. J. Math. **3**, 107–112 (1926)], and van Kampen [Amer. J. Math. **63**, 371–376 (1941); these Rev. **2**, 289]. *N. Levinson.*

Wintner, Aurel. On the convergence of successive approximations. Amer. J. Math. **68**, 13–19 (1946). [MF 15485]

The author shows that the Osgood criterion for uniqueness of the solutions of a system of differential equations is also sufficient to ensure the convergence of the Picard successive approximations. *N. Levinson.*

Wintner, Aurel. The infinities in the non-local existence problem of ordinary differential equations. Amer. J. Math. **68**, 173–178 (1946). [MF 15499]

The author proves that, if the real function $f(t,x)$ is continuous for $0\leqq t\leqq a$, $-\infty<x<\infty$, and if $x=x(t)$, $0\leqq t<t_0$, where $t_0\leqq a$, is a solution of the differential equation $x'=f(t,x)$, then either $x(t_0-0)$ exists as a finite limit, in which case $x(t)$ can be extended to the closed interval $0\leqq t\leqq t_0$ (and if $t_0\neq a$ to an interval $0\leqq t\leqq t_1$ where $t_0<t_1<a$) in such a way as to satisfy the differential equation; or else $x(t_0-0)$ is either $+\infty$ or $-\infty$. *N. Levinson.*

Wintner, Aurel. Asymptotic equilibria. Amer. J. Math. **68**, 125–132 (1946). [MF 15494]

The author shows that his result [same J. **67**, 417–430 (1945), in particular, p. 423; these Rev. **7**, 117] on the asymptotic behavior of solutions of linear differential equations, which states that, if $f_{ik}(t)$, $i,k=1,\cdots,n$, are continuous functions for $0\leqq t<\infty$ such that $\int^\infty|f_{ik}(t)|dt<\infty$,

then every vector solution $(x_1(t), \cdots, x_n(t))$ of the n differential equations $x_i' = f_{i1}(t)x_1 + \cdots + f_{in}(t)x_n$ tends to a finite vector as $t \to \infty$, can be generalized for a large class of nonlinear differential equations. If the continuous functions $f_i(t; x_1, \cdots, x_n)$, $i = 1, \cdots, n$, are such that

$$\int^{\infty} |f_i(t; 0, 0, \cdots, 0)|\, dt < \infty$$

and if there exist nonnegative measurable functions $\lambda_{ik} = \lambda_{ik}(t)$, $0 \leqq t < \infty$; $i, k = 1, \cdots, n$, bounded on any finite interval, such that $\int^{\infty} \lambda_{ik}(t)\, dt < \infty$ and such that

$$|f_i(t; x_1^1, \cdots, x_n^1) - f_i(t; x_1^2, \cdots, x_n^2)| \leqq \sum_{k=1}^{n} \lambda_{ik}(t)\, |x_k^1 - x_k^2|,$$

then any vector solution $(x_1(t), \cdots, x_n(t))$ of the differential equations $x_i' = f_i(t; x_1, \cdots, x_n)$ can be extended over the entire half-axis $0 \leqq t < \infty$ and tends to a finite limiting vector $(x_1(\infty), \cdots, x_n(\infty))$ as $t \to \infty$. The proof depends on the method of successive approximations and shows that $\int^{\infty} |x_i'(t)|\, dt < \infty$, $i = 1, \cdots, n$. A problem of stability raised by the author in the paper is answered by Siegel in an addendum to the effect that the mapping of the initial position $x_1(0), \cdots, x_n(0)$ on the limiting position $x_1(\infty), \cdots, x_n(\infty)$ is continuous and one-to-one. *P. Hartman.*

Malkin, I. Stability in the case of constantly acting disturbances. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 241–245 (1944). (Russian. English summary) [MF 11597]

The author considers the motion given by the differential equations $dx_s/dt = X_s(t, x_1, \cdots, x_n)$, $s = 1, \cdots, n$, where $X_s(t, 0, \cdots, 0) = 0$. The trivial solution $x_s = 0$ is the unperturbed motion. Generalizing the work of Liapounoff, the author gives a definition of stability covering a perturbation of the initial conditions and of the differential equations. A sufficient condition for stability is formulated and proved. In special cases, necessary and sufficient conditions are obtained. *A. Weinstein* (Toronto, Ont.).

Shtokalo, I. Généralisation de la formule fondamentale de la méthode symbolique pour le cas des équations différentielles à coefficients variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 10–11 (1945). [MF 13750]

This is a note on the system of differential equations

$$dx/dt = A(t)x + ce^{pt} \tag{1}$$

in which t is real, $A(t)$ is a square matrix of order n, x and c are vectors and p is a complex scalar. The roots of the determinant equation $|pE - A_0| = 0$, in which E is the unit matrix, are denoted by s_k and $d(p) = \min |\Re(s_k - p)| \geqq \alpha$. The following theorem is announced. To the arbitrary positive numbers N and α there corresponds a positive K such that a system

$$dx/dt = (A_0 + \epsilon f(t))x + ce^{pt}, \tag{2}$$

in which $|f(t)| \leqq N$ over the entire axis of reals, and $|\epsilon| \leqq d(p)/K$, possesses a unique solution $x = \xi(t, p, \epsilon)e^{pt}$, in which $\xi(t, p, \epsilon)$ is bounded as to t and is analytic in p and ϵ. If $f(t)$ is almost periodic, $\xi(t, p, \epsilon)$ is likewise so. A similar theorem is stated for the equation (1). *R. E. Langer.*

Chazy, Jean. Sur les courbes définies par les équations différentielles. C. R. Acad. Sci. Paris **221**, 457–459 (1945). [MF 14686]

This note is concerned with the differential equations

$$dv/d\alpha = v\{\varphi(v) + \sin\alpha\}/\cos\alpha,$$
$$du/d\alpha = 2u(u + \sin\alpha)/(-u + \cos\alpha).$$

Here $\varphi(v)$ is a given function, subject to some stated restrictions. The first equation arises in connection with a simple problem in ballistics; the second equation arises, in a similar manner, in connection with a problem concerning a ski jump. The singular points of the families of solutions of the equations are determined, as well as their natures, and there is some discussion of the physical implications of the results. *L. A. MacColl* (New York, N. Y.).

Niemytzki, V. Intégration qualitative du système d'équations différentielles $dx/dt = Q(x, y)$; $dy/dt = P(x, y)$. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 307–344 (1945). (Russian. French summary) [MF 14586]

The author considers criteria for the density and fineness of approximating polygonal curves to the solutions of the system in order that a complete qualitative picture of the solutions should be determined. The precise statement of his results is lengthy and is given in the seven page French summary. *N. Levinson* (Cambridge, Mass.).

Malmquist, J. Sur les fonctions à un nombre fini de branches satisfaisant à une équation différentielle du premier ordre. Acta Math. **74**, 175–196 (1941). [MF 13769]

The author studies solutions of a differential equation of the form

$$F_0(y; x)y'^m + F_1(y; x)y'^{m-1} + \cdots + F_m(y; x) = 0, \tag{1}$$

where the F's are polynomials in y with coefficients which are algebraic functions of x. After some preliminary discussion concerning the singularities of the solution, the greater part of the paper is devoted to a proof of the following theorem. Suppose that the differential equation is not an equation with fixed critical points, and that there is a solution having a branch, defined in the neighborhood of $x = 0$, which has a finite number of values and an infinity of critical points. Then the equation can be transformed into a Riccati equation $dz/dx = a(x)z^2 + b(x)z + c(x)$ by a transformation $y^n + R_1(z; x)y^{n-1} + \cdots + R_n(z; x) = 0$, or into an elliptic differential equation $(dz/dx)^2 = a(x)(4z^3 - g_2z - g_3)$ by a transformation

$$y^n + R_1(z; x)y^{n-1} + \cdots + R_n(z; x) + (dz/dx)\{\bar{R}_1(z; x)y^{n-1} + \cdots + \bar{R}_n(z; x)\} = 0.$$

Inversely, we have $z = R(dy/dx, y; x)$. The R_i's and $\bar{R}_i$'s are rational functions of z, and R is a rational function of dy/dx and y. The coefficients of these functions, and also a, b and c, are rational functions of x and t, the coefficients in equation (1) being expressed as rational functions of x and a variable t which is related to x by an irreducible algebraic equation. *L. A. MacColl* (New York, N. Y.).

Sezawa, Katsutada, and Utida, Ikuo. On the phenomena of instability in undamped quasi-harmonic vibration. I. Proc. Imp. Acad. Tokyo **19**, 646–652 (1943). [MF 14863]

Undamped vibrations of a system with periodically varying elasticity, or inertia, are governed by differential equations of the form

$$\frac{d}{dt}\left\{P(t)\frac{dX}{dt}\right\} + Q(t)X = 0, \tag{1}$$

where P and Q are functions of t. In particular, there are three cases of practical importance: (a) P = constant, $Q = Q_0 + Q_1 \cos 2pt$; (b) P = constant, $Q = (R_0 + R_1 \cos 2pt)^{-1}$; (c) $P = P_0 + P_1 \cos 2pt$, Q = constant. The first case is that of the Mathieu equation, which has been investigated extensively. The other two are reduced to a common form,

$$d^2X/dy^2 + k\,\mathrm{sn}(y, k)\, dX/dt + \omega^2 X = 0, \tag{2}$$

where in case (b), for example, $k=R_0/R_1-\{(R_0/R_1)^2-1\}^{\frac{1}{2}}$, $\omega^2=k/(2p^2PR_1)$, and

$$y=\{(R_0+R_1)^{\frac{1}{2}}+(R_0-R_1)^{\frac{1}{2}}\}\int_0^{pt}(R_0+R_1\cos 2\tau)^{-\frac{1}{2}}d\tau.$$

In (2), sn (y, k) is replaced by a "square pulse," equal to 1 where sn (y, k) is positive and equal to -1 where sn (y, k) is negative. The equation so obtained can be integrated exactly, and for this approximate equation the "stability chart" is constructed and the vibrations are discussed. There is no attempt to show that the approximation in (2) of sn (y, k) by a square pulse leads to better results than the approximation in (1) of $\cos 2pt$ by a square pulse.

A. Erdélyi (Edinburgh).

Sezawa, Katsutada, and Utida, Ikuo. On the phenomena of instability in undamped quasi-harmonic vibration. II. Proc. Imp. Acad. Tokyo **20**, 128–132 (1944). [MF 14879]

[Cf. the preceding review.] The authors investigate vibrations of a model propeller which is capable of a tilting motion governed by the (finite) resistance of the elastic force of the engine mounting. Under the assumption that the tilting motion takes place in a fixed vertical plane, the differential equation of the vibrations is of the form discussed in part I; it corresponds to case (c). Experimental results show instability phenomena which are in qualitative agreement with the mathematical theory of part I. No attempt is made at a detailed quantitative comparison.

A. Erdélyi (Edinburgh).

Abelé, Jean. Définition cinématique d'oscillations de relaxation discontinues. C. R. Acad. Sci. Paris **221**, 656–658 (1945). [MF 15153]

Several choices for f are made, where $z=\cos\theta-f(\sin\theta)$ and the motion of θ given by

$$wt=-\int_{\theta_0}^{\theta}\sin\theta(d\theta/dz)d\theta$$

is considered. [Cf. J. Phys. Radium (8) **6**, 96–103 (1945); these Rev. **7**, 158.] *N. Levinson* (Cambridge, Mass.).

Haag, J. Étude asymptotique des oscillations de relaxation. Ann. Sci. École Norm. Sup. (3) **60**, 35–64, 65–111 (1943). [MF 14638]

Haag, J. Exemples concrets d'étude asymptotique d'oscillations de relaxation. Ann. Sci. École Norm. Sup. (3) **61**, 73–117 (1944). [MF 14646]

The author studies in great detail the problem of computing the trajectories in the phase plane and the period of closed trajectories for the equations

$$dx/dt=(\omega/\epsilon)y-(1/m\epsilon)F(x),\quad dy/dt=-\epsilon\omega x,$$

where ω and m are constants, ϵ is a very small constant, xF is negative for small x and positive for large x, $F(0)=0$ and $F(x)$ is increasing (decreasing) for large positive (negative) x. These equations are the Liénard form of the equation $em\ddot{x}+F'(x)x\dot{x}+em\omega^2x=0$. The author shows the utility of his technique in the second paper by carrying out calculations for a number of interesting examples including the van der Pol equation. *N. Levinson* (Cambridge, Mass.).

Haag, Jules. Sur la théorie de la synchronisation. C. R. Acad. Sci. Paris **221**, 682–684 (1945). [MF 15156]

This note is concerned with the differential equation

$$\ddot{\theta}+\omega^2\theta=\lambda K(t,\theta,\dot{\theta}),$$

where ω and λ are constants and K is periodic with respect to t, with a period T which is not an integral multiple of $2\pi/\omega$. The following theorem is proved. If $K(t, 0, 0)\not\equiv 0$, if, when $|\theta|<a$ and $|\dot{\theta}|<b$, K is bounded and satisfies a Lipschitz condition with respect to θ and $\dot{\theta}$, and if λ is sufficiently small in absolute value, the differential equation possesses a unique periodic solution with the period T. Some discussion is given of the case in which $\omega T/(2\pi)$ is near an integer. In this case the phenomena of resonance appear.

L. A. MacColl (New York, N. Y.).

Haag, Jules. Sur l'amortissement et l'entretien des oscillations à n degrés de liberté. C. R. Acad. Sci. Paris **221**, 734–736 (1945). [MF 15162]

The author considers a linear dynamical system for which the kinetic energy T, the potential energy V and the dissipation function S have the special forms

$$2T=M_0\dot{x}_0^2+\sum_{i=1}^{n}(M_i\dot{x}_i^2+2N_i\dot{x}_0\dot{x}_i),$$

$$V=\sum_{i=0}^{n}M_i\omega_i^2x_i^2,\quad S=\sum_{i=0}^{n}\lambda_iM_i\omega_ix_i^2,$$

the λ's being small positive constants (x_0 may be the coordinate of a body S_0, and x_i $(i>0)$ may be the coordinate of a body S_i relative to S_0). Some properties of the free vibrations of the system are discussed. Subsequently it is assumed that S_1 is subjected to an impulse each time that x_1 passes through the value zero, each impulse communicating energy in the amount E to the system; E is determined so that S_1 shall perform, with respect to S_0, a periodic oscillation with a prescribed amplitude. *L. A. MacColl.*

Teodorčik, K. Energetics of asynchronous reactions. Uchenye Zapiski Moskov. Gos. Univ. Fizika **77**, 112–116 (1945). (Russian) [MF 15111]

In the case of the van der Pol equation with a forcing term an approximate solution is obtained consisting of the sum of a sinusoidal term of the same frequency as the forcing term and a sinusoidal term associated with self-excited oscillations of the unforced system. Another equation is also considered. *N. Levinson.*

Sadowsky, Michael A. Nonlinear springs. J. Franklin Inst. **240**, 469–476 (1945). [MF 14547]

The equation $\ddot{y}+S(y)=0$, where $S(y)$ is an odd function, is considered. The solution $y(t, v)$ corresponding to $\dot{y}=v$ when $y=0$ can always be written as an integral. In case this integral is not tractable the author introduces the device of expressing not y but some known function $G(y)$ as a function of t and v. He develops this device as a useful method. *N. Levinson* (Cambridge, Mass.).

*MacColl, LeRoy A. **Fundamental Theory of Servomechanisms.** D. Van Nostrand Co., Inc., New York, 1945. xviii+130 pp. $2.25.

This introduction to the theory of servomechanisms is designed in part to attract the attention of mathematicians to the problems of automatic control. Its emphasis is upon mathematical questions rather than practical details. The author treats the servomechanism as a feedback amplifier. This enables him to develop the subject in terms of concepts familiar to communication engineers. Thus properties of linear servomechanisms are analyzed by studying their characteristics in the complex frequency plane. The important question of stability is discussed by use of the Nyquist criterion, preferred by the writer to the Routh rule used by

others. It is shown how the performance of a servomechanism may be analyzed by its response to a standard input signal. The general principles are illustrated by detailed examples, for instance, the problem of stabilizing a steering system.

The viewpoint adopted in this book is different from most expositions in that the analysis emphasizes the steady-state theory rather than the transient responses. A chapter is devoted to justifying this approach, and the connections between the two are discussed briefly. The remainder of the subject matter consists of a treatment of servomechanisms with discrete rather than continuous inputs (the sampling mechanism), and a discussion of the nonlinear on-off servomechanism. There are a foreword by Warren Weaver and a bibliography. *H. Pollard and B. Yood.*

Lourie, A. I. Stability of one type of systems under control. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 353–367 (1945). (Russian. English summary) [MF 15347]

The stability of the system

$$\dot{x}_k = -\rho_k x_k + f(\sigma), \qquad k = 1, 2, \cdots, n,$$
$$\dot{\sigma} = \beta_1 x_1 + \cdots + \beta_n x_n - f(\sigma)$$

is considered for large displacements from the equilibrium position $x_k = 0$, $\sigma = 0$. An appropriate, but not obvious, Liapounov function V is introduced. Stability is studied using V and $\dot{V}$. Applications are made to specific automatic control devices. *N. Levinson* (Cambridge, Mass.).

Andronow, A., et Mayer, A. Le problème de Wischnegradski dans la théorie de la régulation directe. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 340–343 (1945). [MF 14020]

The nonlinear problem of Wischnegradski is solved by the authors through the use of a device similar to that by which they solved the governor problem of von Mises [C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 54–58 (1944); these Rev. **6**, 154]. Here again the phase space is 3-dimensional but by an ingenious device the problem is reduced to that of studying a transformation of the plane into itself.

N. Levinson (Cambridge, Mass.).

Andronow, A., Bautin, N., et Gorelik, G. Auto-oscillations d'un schéma simplifié contenant une hélice à pas variable automatique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 263–267 (1945). [MF 14028]

Considering the system $\dot{x} = f(x, y, z)$, $\dot{y} = z$, $\dot{z} = g(x) - z$ in three-dimensional space, $P(x, y, z)$ is either between the planes $y - x = \pm\frac{1}{2}$ or in one of the half-planes (H) $x - y = \frac{1}{2}$, $z < 0$, or (H') $x - y = -\frac{1}{2}$, $z > 0$. Between the planes, $f = 0$, while on H and H', $f = z$. The function $g(x)$ is odd. Moreover, $g(x) = 0$, $0 \leqq x \leqq \psi_0$; $g(x) = -\alpha(x - \psi_0)$, $\psi_0 \leqq x \leqq \psi_1$; $g(x) = -\alpha(\psi_1 - \psi_0)$, $x > \psi_1$. Using the system to define certain geometric transformations, a complete solution of steady state behavior of the system including the question of stability is given. *N. Levinson* (Cambridge, Mass.).

Bateman, H. The control of an elastic fluid. Bull. Amer. Math. Soc. **51**, 601–646 (1945). [MF 13337]

This is the 1943 Gibbs lecture. If $\sum_0^n a_i D^i(y) = f(t)$ describes a regulator mechanism, then a sufficient condition for stability is that the roots of the characteristic equation have negative real parts. Several criteria for this are discussed. The case of time lag in the restoring forces has been treated by (a) expanding $f(t-\alpha)$ by Taylor's formula, (b) differential difference equations, (c) Laplace transform theory. Typical examples are $x'' + Qx' + Nf'(t-h) + Pf(t-h) = 0$, with N and P constant, or the equation of Volterra type, $x(t) - y(t) + \lambda \int_0^t k(x - T)x(T)dT = 0$. The section on acoustics involves several interesting physical interpretations of the second order constant coefficient ordinary differential equation connected with propagation of sound through partitions. The problem of normal passage of plane waves through a series of equivalent plates leads to a pair of functional equations first given by Stokes. The problems of wave filters are represented by a résumé of Lamb's treatment of the string loaded at regular intervals with equivalent complexes of springs. Rayleigh's reaction theorem is stated together with generalizations due to the author and others. The report restricts itself to linear or linearized situations treated by classical methods. *D. G. Bourgin* (Urbana, Ill.).

Schelkunoff, S. A. Solution of linear and slightly nonlinear differential equations. Quart. Appl. Math. **3**, 348–355 (1946). [MF 14525]

This paper is concerned primarily with the solution of systems of linear ordinary differential equations with variable coefficients. The author's method of solution consists essentially of taking, as the first approximation to the solution, the solution of a related and simpler system of differential equations satisfying the given initial or boundary conditions, and then obtaining higher approximations by a perturbation method. The related system of differential equations may, for instance, be a system of linear equations with constant coefficients, each constant coefficient being a mean value of the corresponding coefficient in the original system. It is pointed out that the method can be applied, with appropriate modifications, to "slightly nonlinear" equations. By way of illustration, there is a rather extensive discussion of the application of the method to the transmission line equations $dV/dx = -Z(x)I$, $dI/dx = -Y(x)V$. The method is also applied to the solution of Rayleigh's equation for a nonlinear oscillator, $\ddot{q} + (R_1 + R_2\dot{q}^2)\dot{q} + \omega^2 q = 0$.

L. A. MacColl (New York, N. Y.).

Zhabotinsky, M. E. On a particular case of systems with two degrees of freedom. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **15**, 573–586 (1945). [MF 15515]

An approximate treatment is given for the system

$$\ddot{x} + x = \epsilon^\lambda f(x, \dot{x}, y, \dot{y}, \epsilon),$$
$$\ddot{y} + y = \epsilon^\gamma g(x, \dot{x}, y, \dot{y}, \epsilon),$$

where λ and γ are not necessarily equal. Approximate solutions of periodic form are found and their stability studied using results of Poincaré and Liapounov. *N. Levinson.*

Četajev, N. G. The smallest characteristic number. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 193–196 (1945). (Russian. English summary) [MF 14031]

In the system (1) $dx_s/dt = \sum_{r=1}^n p_{sr}x_r$, $s = 1, \cdots, n$, let p_{sr} be continuous real functions of t for $t > 0$. The author proves the following theorem. "If, as t approaches infinity, the coefficients p_{sr} tend to the definite limits c_{sr} then the smallest characteristic number of the system (1) coincides with the smallest characteristic number of the limit system," with constant coefficients. [The author is apparently unfamiliar with the work of O. Perron [Math. Z. **1**, 27–43 (1918) and earlier papers], where much more extensive results are obtained for nth order linear differential equations.]

N. Levinson (Cambridge, Mass.).

*Golubev, V. V. **Lekcii po Analitičeskoĭ Teorii Diferencial'nyh Uravneniĭ. [Lectures on the Analytic Theory of Differential Equations].** Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1941. 398 pp. (Russian)

(I) Existence theorems. Uniqueness of solutions. Singular points. (II) Equations of the first order. Elements of the theory of algebraic functions. (III) Equations of the second order and nonmovable critical points. (IV) Linear equations. (V) Hypergeometric functions. Riemann's problem. (VI) Reflection of polygons bounded by circular arcs. (VII) Elements of the theory of automorphic functions. (VIII) Automorphic functions of Fuchs and Klein.

Table of contents.

Arley, Niels. On linear differential equations of infinitely high order. Mat. Tidsskr. B. 1945, 71–74 (1945). (Danish) [MF 14262]

Consider the differential equation of infinite order

$$\sum_n \alpha_n(x)y^{(n)}(x)=\beta(x).$$

Putting $y_k=y^{(k)}$, this equation reduces to a system of equations of first order of a type which has been studied recently in connection with stochastic processes [Arley and Borchsenius, Acta Math. 76, 261–322 (1945); these Rev. 7, 161].

W. Feller (Ithaca, N. Y.).

De Donder, Th., et Géhéniau, J. Théorie variationnelle des perturbations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 637–644 (1943). [MF 13859]

On suppose donné un système d'équations aux dérivées partielles $E_\alpha=0$ à m fonctions inconnues y^α, dont les premiers membres contiennent un paramètre σ, par rapport auquel on peut les développer en séries de Maclaurin, aussi bien que les fonctions y^α. Le système proposé se transforme alors en un système infini d'équations aux dérivées partielles aux m fonctions y^α et à leurs dérivées successives par rapport à σ, pour $\sigma=0$. On applique ensuite le théorème au système d'équations lagrangiennes équivalent à la condition de minimum d'une intégrale qui dépend de σ et d'un paramètre indépendant τ. On conclut que cette condition équivaut à un système de conditions analogues relatives aux dérivées successives, par rapport à σ, pour $\sigma=0$, de l'intégrale considérée.

B. Levi (Rosario).

Gavrilov, A. F. Application of the method of small parameters to the integration of certain partial differential equations (nonlinear generalizations of the telegraph equation). Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 13, 55–78 (1944). (Russian. Georgian summary) [MF 14620]

The author considers in some detail the long line of finite length with L, C, and G constants but with $R(x, t)$ satisfying the integral equation

$$(*) \qquad R=R_0+\epsilon\int_0^t i^2R\,dt,$$

where $i(x, t)$ is current. The occurrence of i^2 as well as R under the integral sign introduces nonlinearity. Using ϵ as a small parameter the long line equations are solved subject to general initial and boundary conditions. The equation (*) takes account of changes in resistance due to heating of the line. The practical value of the result is not clear.

N. Levinson (Cambridge, Mass.).

Fédoroff, V. Sur la monogénéité dans l'espace. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 257–274 (1945). (Russian. French summary) [MF 15357]

[An abstract appeared in C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 222–223 (1945); these Rev. 7, 149.] Let $u(x, y, z)$, $v(x, y, z)$ be of class C'' and satisfy $\nabla u\times\nabla v\neq 0$ in an (x, y, z)-domain D. If for each point of D there exists a vector function $\mathbf{a}$ such that $\nabla u=\nabla v\times\mathbf{a}$, $\nabla v=\mathbf{a}\times\nabla u$, the author writes (1) $u+iv\sim\mathbf{a}$. Necessary and sufficient conditions for (1) are (2) $\nabla u\cdot\nabla v=0$, $(\nabla u)^2=(\nabla v)^2\neq 0$. It is remarked that functions u, v satisfying (1) might or might not be harmonic. If (1) holds, then for every function $f(\zeta)$, $\zeta=u+iv$, analytic in ζ and satisfying $f'(\zeta)\neq 0$ for (x, y, z) in D, we have $f(\zeta)\sim\mathbf{a}$, $f(\bar{\zeta})\sim-\mathbf{a}$. If both $u+iv$ and $P+iQ$ satisfy (1), then to each point p of D there corresponds a neighborhood of p in which $P+iQ$ is an analytic function, with nonvanishing derivative, of $u+iv$. For sufficiently smooth functions φ, the classes of functions u, v, φ satisfying $u+iv\sim\nabla\varphi$ are determined. Finally, in generalizing equations (2), the author considers functions $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ satisfying

$$(3) \qquad \nabla u\cdot\nabla v=\nabla v\cdot\nabla w=\nabla w\cdot\nabla u=0, \quad (\nabla u)^2=(\nabla v)^2=(\nabla w)^2\neq 0,$$

and shows that the only harmonic solutions of (3) are linear functions of x, y, z.

[We observe that sets of harmonic functions satisfying (2) or (3) are special cases of sets of functions $u_j(x_1, \cdots, x_n)$, $j=1, \cdots, m$, harmonic in a domain and satisfying

$$(4) \qquad \sum_{i=1}^{n}\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_l}=[\lambda(x_1, \cdots, x_n)]\delta_{k,l}, \quad k, l=1, 2, \cdots, n,$$

where $\delta_{k,l}$ is the Kronecker delta. Sets satisfying (4), called sets of conjugate harmonic functions, have previously been studied, for $n>2$, by Cioranesco [Bull. Sci. Math. (2) 56, 55–64 (1932)], who considered only the case $m=n$; by the reviewer [Duke Math. J. 10, 335–339 (1943); these Rev. 4, 278], who made the erroneous remark that the study of these functions is equivalent to the study of minimal varieties in conformal representation; and by the reviewer and R. H. Bing [Duke Math. J. 10, 637–640 (1943); these Rev. 5, 158], who showed that for $n>2$ the only sets of conjugate harmonic functions are sets of linear functions. This last result contains as a special case the final result of the present paper.]

E. F. Beckenbach (Los Angeles, Calif.).

Brelot, Marcel. Sur l'allure des fonctions harmoniques à la frontière. C. R. Acad. Sci. Paris 220, 676–678 (1945). [MF 14063]

The author indicates two extensions of theorems on irregular boundary points proved earlier [Bull. Sci. Math. (2) 68, 12–36 (1944); these Rev. 7, 15].

L. H. Loomis.

Inoue, Masao. Sur l'approximation de la solution du problème de Dirichlet. Proc. Imp. Acad. Tokyo 19, 431–437 (1943). [MF 14839]

Many authors have studied the approximate solutions of boundary value problems for partial differential equations obtained by replacing the differential equation by a difference equation involving values of the unknown function at nodes of a suitably chosen network. [See, for example, J. Le Roux, J. Math. Pures Appl. (6) 10, 189–230 (1914); H. Liebmann, S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1918, 385–416.] The author sketches a method of establishing the convergence of the approximation to the solution

of the differential equation when the number of nodes becomes infinite. *F. W. Perkins* (Hanover, N. H.).

Liénard, Alfred. Problème de la dérivée oblique dans la théorie du potentiel. C. R. Acad. Sci. Paris **220**, 427–428 (1945). [MF 13956]

The author is concerned with the boundary value problems for harmonic functions V in three-space, in which the boundary conditions are (1) $\partial V/\partial\sigma+c(M)V=f(M)$, M being a point of the boundary surface S, σ a direction dependent on M, and $c(M)$, $f(M)$ given functions. He shows that any integral condition which $f(M)$ must satisfy in order to guarantee a solution of (1) can be obtained by considering an associated homogeneous boundary value problem (2) $\partial W/\partial\sigma'+c'(M)W=0$, in which the direction σ' and the function c' are suitably defined in terms of σ and c. If u is the angle between σ and the interior normal to S and W is a solution of (2), the condition $\int f(M)W\sec u\,dS=0$ is necessary for (1) to have a solution. Conversely, if $\int f(M)F(M)dS=0$ is a necessary condition for (1) to have a solution, then $F(M)/\cos u$ are the boundary values of a harmonic function satisfying (2). The proofs are by means of the Green identities. *J. W. Green* (Los Angeles, Calif.).

Deny, Jacques. Distributions conjuguées. Applications aux fonctions d'une variable complexe et aux séries de Fourier. C. R. Acad. Sci. Paris **221**, 532–534 (1945). [MF 15141]

Let E be a closed bounded set in the plane and let E^* be the complement of the unbounded component of CE. Let $U^\lambda(z)$ be the value at z of the logarithmic potential of the distribution λ on E. Two distributions λ and μ on E are said to be conjugates provided $U^\lambda-iU^\mu$ is analytic in CE^*; for this, it is necessary and sufficient that $\int z^n(d\lambda+id\mu)=0$, $n=0, 1, 2, \cdots$. Thus, if E is a simple closed rectifiable curve, the distributions λ and μ defined on E by $d\lambda+id\mu=dz$ are conjugates. If E has no interior points and coincides with E^*, then E admits no couples of conjugate distributions. Otherwise E admits an infinitude of such couples; the total masses $\lambda(E)$ and $\mu(E)$ necessarily are null.

The following extension of the Cauchy integral theorem is given. If $F(z)$ is holomorphic in a domain D and if λ and μ are conjugate distributions on a closed and bounded set E such that $E^*\subset D$, then $\int F(d\lambda+id\mu)=0$. A similar extension of the theorem of Morera is given. *E. F. Beckenbach.*

Yosida, Yôiti. Sur le principe du maximum dans la théorie du potentiel. Proc. Imp. Acad. Tokyo **17**, 476–478 (1941). [MF 14730]

Given a positive mass distribution μ on a bounded closed set F of space, let

$$u(P)=\int_F r_{QP}^{-a}d\mu_Q, \qquad 1\leqq a\leqq 3.$$

The author cites a theorem due to Frostman [Comm. Sém. Math. Univ. Lund. [Medd. Lunds Univ. Mat. Sem.] **3**, 1935] which asserts that, if $u(P)\leqq M$ everywhere on F, then $u(P)\leqq M$ throughout space. The paper contains a simplified treatment of this theorem, based directly on properties of integrals. *F. W. Perkins* (Hanover, N. H.).

Beckenbach, E. F., and Reade, Maxwell. Regular solids and harmonic polynomials. Duke Math. J. **12**, 629–644 (1945). [MF 15506]

Pour les polyèdres réguliers (tétraèdre, cube, octaèdre, dodécaèdre, icosaèdre), on cherche à caractériser les fonctions continues dont la valeur au centre soit égale à la moyenne des valeurs aux sommets pour certaines familles de ces polyèdres. On donne diverses conditions équivalentes et on trouve comme solutions certains polynômes harmoniques. Cette étude, à peu près proposée par Walsh qui avait traité les problèmes plans analogues [Bull. Amer. Math. Soc. **42**, 923–930 (1936)] est en relation étroite et qu'il faudrait approfondir avec les recherches générales de Choquet-Deny et Schwartz [Bull. Soc. Math. France **72**, 118–140, 141–145 (1944); ces Rev. **7**, 161, 156]. *M. Brelot* (Grenoble).

Gavrilov, L. Solution of the direct and inverse potential problem for the paraboloid of rotation and the infinite parabolic cylinder. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **9**, 521–528 (1945). (Russian. English summary) [MF 15638]

In the present paper author considers the solution of the direct problem of the gravity potential for the paraboloid of rotation and the parabolic cylinder passing to the limit from the ellipsoid and elliptic cylinder. Maclaurin's theorem of confocal ellipsoids and elliptic cylinders may be extended to paraboloids and parabolic cylinders. The inverse problem is not yet solved for all the cases which here arise. *Author's summary.*

Agostinelli, Cataldo. Su di una proprietà del momento magnetico di una sfera. Boll. Un. Mat. Ital. (2) **5**, 1–5 (1943). [MF 16077]

Si dimostra che il momento magnetico di una sfera magnetizzata parallelamente a un suo diametro e che non abbia poli sulla sua superficie è equivalente al momento magnetico di un magnete elementare posto nel suo centro e coll'asse coincidente con quel diametro. *Author's summary.*

Teissier du Cros, F. Sur une propriété des fonctions biharmoniques. C. R. Acad. Sci. Paris **218**, 957–959 (1944). [MF 15244]

If U, T, S are concentric spheres in 3-space, U in T and T in S, and if a biharmonic function in the closure of S is nonnegative in U and between T and S, nonpositive between U and T, and 0 on U and T, then it vanishes identically. *S. Bochner* (Princeton, N. J.).

Teissier du Cros, F. Sur l'ensemble des fonctions biharmoniques, régulières dans un cercle. C. R. Acad. Sci. Paris **219**, 355–357 (1944). [MF 15264]

The author has already shown that certain simple properties serve to characterize the set of all functions harmonic in the interior of a given circle [same C. R. **216**, 437–438 (1943); these Rev. **6**, 146]. He now gives a similar characterization of the set of all functions biharmonic and regular in a circle C. Suppose that, for some constants α and β and for every set of three concentric circles c_1, c_2 and c_3 interior to C, the expression $f(x, y)-\alpha-\beta(x^2+y^2)$ does not assume values of a certain sign on the median circumference c_2 and does not assume the opposite sign on the circumferences c_1 and c_3. If this hypothesis implies that $f(x, y)=\alpha+\beta(x^2+y^2)$ then $f(x, y)$ is said to have the property P''. Let (x, y) be transformed into (x', y') by an inversion or reflection which carries a circle γ interior to C into γ'. Let $\gamma(x, y)$ and $\gamma'(x', y')$ be the powers of (x, y) and (x', y') with reference to the circles γ and γ', respectively. The author defines a transformation T carrying $f(x, y)$ into $f'(x', y')$ by setting $f'(x', y')=f(x, y)\gamma'(x', y')/\gamma(x, y)$. An algebraically additive

set of functions having the property P'' which admits the transformations T contains only biharmonic functions, and the set of all biharmonic functions meets these requirements.
F. W. Perkins (Hanover, N. H.).

Vekua, I. N. Sur un certain développement des fonctions métaharmoniques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 3–6 (1945). [MF 15224]

It has been shown by Cioranescu [Bull. Soc. Math. France **65**, 41–52 (1937)] that a real function $u(x_1, \cdots, x_p)$, analytic in a neighborhood of a point $P_0: (x_1^0, \cdots, x_p^0)$, can be developed in a series

$$\sum_{k=0}^{\infty} u_k(x_1, \cdots, x_p) r^{2k}, \qquad r^2 = \sum_{j=1}^{p} (x_j - x_j^0)^2,$$

where the u_k are harmonic at P_0; the series converges absolutely and uniformly in a sufficiently small neighborhood of P_0. For metaharmonic functions u, the author exhibits formulas for the harmonic functions u_k and obtains more precise results concerning the domain of convergence of the series. *E. F. Beckenbach* (Los Angeles, Calif.).

Kodaira, Kunihiko. Über die Rand- und Eigenwertprobleme der linearen elliptischen Differentialgleichungen zweiter Ordnung. Proc. Imp. Acad. Tokyo **20**, 262–268 (1944). [MF 14891]

The author handles linear elliptic self-adjoint partial differential equations in n variables using the theory of operators in Hilbert space and Weyl's method of orthogonal projection. The operator $L(u)$ is the Euler operator for a quadratic integral of the form $(au, au) = \text{minimum}$, where au is the "scalar-vector" $(u, \partial u/\partial x_1, \cdots, \partial u/\partial x_n)$ and $(\Phi, \Psi) = \int_G (a_{\alpha\beta}\varphi_\alpha\psi_\beta + \varphi b_\alpha \psi_\alpha + \psi b_\alpha \varphi_\alpha + c\varphi\psi)dx$, Φ, Ψ being scalar-vectors in the space L_2 for which (Φ, Φ) and $(\varphi, \varphi) < +\infty$; l_2 denotes the ordinary l_2 space for scalar functions on G. The functions $a_{\alpha\beta}$, b_α, c are all supposed to be "regular," that is, to be sufficiently differentiable, and to satisfy an inequality condition $a_{\alpha\beta}\varphi_\alpha\varphi_\beta + 2b_\alpha\varphi_\alpha\varphi + c\varphi^2 \geqq k\varphi_\alpha\varphi_\alpha$ for all x on G, all real scalar-vectors Φ and some $k > 0$. The space L_2 is complete with respect to the norm $\{Q(\Phi, \Phi)\}^{\frac{1}{2}}$, where $Q(\Phi, \Phi) = (\Phi, \Phi) + (\varphi, \varphi)$. The manifold N is the Q-closure in L_2 of the set of vectors au for which u is of class C''' and vanishes on and near the boundary G^* of G; ν is the manifold in l_2 of functions u for which $au \varepsilon N$. The ordinary boundary value problem for $L(u) = 0$ is stated as follows: given a bounded differentiable function f such that af is in L_2, to find a regular function φ for which $L(\varphi) = 0$ and $\varphi - f$ is in ν. The existence of φ is found by Weyl's method as follows. (1) In the manifold N, the norms $(\Phi, \Phi)^{\frac{1}{2}}$ and $\{Q(\Phi, \Phi)\}^{\frac{1}{2}}$ are equivalent. (2) Thus af may be written as the sum of two scalar-vectors Φ and Z such that Z is in N and $(\Phi, Y) = 0$ for all Y in N. (3) Using a Green's formula, it is found that, if φ and z are the first members of Φ and Z, then $z \varepsilon \nu$ and $(\varphi, L(\eta)) = 0$ for all η of class C''' on G which vanish on and near G^*. (4) Using the general lemma below it follows that φ is regular with $L(\varphi) = 0$.

The general lemma is the following. Let $D(u)$ be any self-adjoint operator of the form $L(u) + hu$, where h is any regular function. Suppose that e is in l_2 and satisfies $(e, D(u)) = 0$ for all u of class C''' on G and vanishing on and near G^*. Then e is (equivalent to) a regular function with $D(e) = 0$ on G. The operator $L(u)$ is extended in a certain way and the existence of isolated characteristic values $\lambda_j \to +\infty$ and the regularity of the characteristic functions is proved using this lemma. The nonhomogeneous equation is treated by similar methods. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Matsumoto, Toshizô. Sur le principe de Duhamel-Nomitsu. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **22**, 381–391 (1939). [MF 15553]

If $F(x, y; \lambda)$ is a solution of (*) $u_{xx} = u_y$ $(x > 0, y > 0)$ and $F(x, 0, \lambda) = 0$ for $x > 0$, $F(0, y, \lambda) = f(\lambda)$ for $y > 0$, then the principle of Duhamel states that the solution of (*) with boundary values $u(x, 0) = 0$, $x > 0$, and $u(0, y) = f(y)$, $y > 0$, is given by

$$u(x, y) = \int_0^y \frac{\partial F(x, y-\lambda; \lambda)}{\partial y} \partial\lambda.$$

The paper gives sufficient conditions on the function F for the principle of Duhamel to hold, and also applies this method of solution to other types of differential equations.
F. G. Dressel (Durham, N. C.).

Parodi, Maurice. Propagation de la chaleur dans un milieu hétérogène. C. R. Acad. Sci. Paris **219**, 177–179 (1944). [MF 15252]

Let θ and P denote temperature and flux of heat in a one-dimensional heterogeneous medium. The author develops a formal solution of the equations $P = K\partial\theta/\partial x$, $\partial P/\partial x = \gamma\partial\theta/\partial t$ by writing $P(x, t) = \sum_i P_i(x, t)$, $\theta(x, t) = \sum \theta_i(x, t)$ and determining P_i and θ_i by a simple recursive process.
C. C. Torrance (Cleveland, Ohio).

Hadamard, J. Problèmes à apparence difficile. Rec. Math. [Mat. Sbornik] N.S. **17(59)**, 3–8 (1945). (French. Russian summary) [MF 14590]

This paper points out that many theorems, such as Picard's theorem, are difficult to prove by the logical direct method and yet are easily proved by an indirect method. The author gives two examples from the theory of linear partial differential equations. (i) Volterra's difficult solution of the equation of cylindrical waves is contrasted with Parseval's simple solution by the method of descent. (ii) The symmetry of the coefficients in Hadamard's elementary solution of a linear partial differential equation is really a property of geodesics in Riemannian space. Yet the only proof is the indirect one which follows when one constructs the elementary solution. *E. T. Copson* (Dundee).

Krasnooshkin, P. The interaction of oscillating systems with distributed parameters. Acad. Sci. USSR. J. Phys. **9**, 439–446 (1945). [MF 15123]

The paper is concerned with the waves travelling in a set of one-dimensional coupled oscillating systems with distributed parameters. The simplest example is that in which two parallel violin strings are coupled by joining corresponding points by light springs. Such a system is a kind of mechanical analogue of a two-dimensional wave guide. The system of differential equations to be solved in the general case is

$$\rho_k \frac{\partial^2 y_k}{\partial t^2} - \alpha_k \frac{\partial^2 y_k}{\partial x^2} + \sum_{j=1}^{n} \lambda_{jk}(y_k - y_j) - \sum_{j=1}^{n} \sigma_{kj} \frac{\partial^2 y_j}{\partial t^2} = 0$$

for $k = 1, 2, \cdots, n$. If there is a solution of the form $y_k(x, t) = q_k(x)e^{i\omega t}$, the system becomes

$$\frac{d}{dx}\frac{\partial G}{\partial \dot{q}_k} - \frac{\partial F}{\partial q_k} = 0, \qquad k = 1, 2, \cdots, n,$$

where $F = \frac{1}{2}\sum b_{jk} q_j q_k$, with $b_{kk} = -\rho_k\omega^2 + \sum_j \lambda_{kj}$, $b_{jk} = \sigma_{kj}\omega^2 - \lambda_{kj}$

($j \neq k$), and $G - \frac{1}{2}\sum \alpha_k \dot{q}_k^2$ with $\dot{q}_k = dq_k/dx$. This is exactly similar to the Lagrange equations for the small oscillations of a dynamical system with n degrees of freedom; it is treated in the usual manner, and various properties of the solution are deduced. *E. T. Copson* (Dundee).

Nef, Walter. Funktionentheorie einer Klasse von hyperbolischen und ultrahyperbolischen Differentialgleichungen zweiter Ordnung. Comment. Math. Helv. **17**, 83–107 (1945).

Fueter, Nef, and others have developed a function theory related to certain types of second order differential equations suggested by the relationship of analytic functions to Laplace's equation [see, for example, R. Fueter, same Comment. **16**, 19–28 (1944); these Rev. **5**, 261]. The author applies this method to hyperbolic and ultrahyperbolic differential equations of the type

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\delta_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k}=0,$$

where the deltas are constants and where the values of $\partial u/\partial x_j$, $j=1, \cdots, n$, are prescribed on an $(n-1)$-dimensional hypersurface. The author relates solutions of this differential equation to a class of regular functions of a suitable hypercomplex variable. *W. T. Martin* (Syracuse, N. Y.).

Chandrasekhar, S. The radiative equilibrium of an expanding planetary nebula. I. Radiation pressure in Lyman-α. Astrophys. J. **102**, 402–428 (1945). [MF 15113]

The author's summary is as follows. "In this paper the problem of the radiative equilibrium in Lyman-α of a differentially expanding planetary nebula is formulated de novo. Particular attention has been given to the formulation of the boundary conditions. It is shown that the transfer problem reduces to a novel boundary-value problem in hyperbolic partial differential equations. Explicit solutions have been found for the case in which the line absorption coefficient $\sigma(\nu)$ has a rectangular form. On the basis of the solutions obtained, the question of the radiation pressure in Lyman-α has been re-examined. It is shown that when the Doppler shift due to the difference in velocities between the inner and the outer boundaries of the nebula exceeds the line width $2\Delta\nu$ of $\sigma(\nu)$ by a factor $2\sqrt{3}$, the nebula can be divided into three parts: an inner, a central, and an outer part. In the central part the radiation pressure in Lyman-α is of the same order as that in the continuum, while in the inner and the outer parts it is not inappreciable. The bearing of this manner in which the radiation pressure in Lyman-α operates in an expanding atmosphere on the dynamics of a planetary nebula and also on other astrophysical problems is briefly indicated."

The boundary value problem to which the author refers is as follows: to find the solution of $\partial^2 f/\partial x^2 - \partial^2 f/\partial y^2 + f = 0$ given (i) the values of f and $\partial f/\partial y$ on $y=0$, $0 \leqq x \leqq x_1$, (ii) the value of $\partial f/\partial x - \partial f/\partial y$ on $x=0$, $0 \leqq y \leqq y_1$, (iii) the value of f on $x=x_1$, $0 \leqq y \leqq y_1$. The method of characteristics of Riemann determines f only in the triangle bounded by $y=0$, $y=x$, $x+y=x_1$. Outside this triangle additional data are required on $x=0$ and on $x=x_1$. The additional data are found by solving two Volterra integral equations which ensure the continuity of the solution across the characteristics $y=x$ and $x+y=x_1$. This method has been previously used by the author in another astrophysical problem [Rev. Modern Phys. **17**, 138–156 (1945); these Rev. **7**, 177].

E. T. Copson (Dundee).

Integral Equations

Michal, Aristotle D. Differential equations in Fréchet differentials occurring in integral equations. Proc. Nat. Acad. Sci. U. S. A. **31**, 252–258 (1945). [MF 13292]

Consider the Volterra integral equation

$$f(x)=y(x)+\int_a^x K(x,\xi)y(\xi)d\xi, \qquad a \leqq x \leqq b,$$

with continuous $K(x, \xi)$ and $f(x)$, and let $k(x, \xi)$ be the resolvent kernel, so that the equation has the unique continuous solution

$$y(x)=f(x)+\int_a^x k(x,\xi)f(\xi)d\xi.$$

The author derives the nonlinear total differential systems of equations satisfied by the Fréchet differentials δk and δy in terms of δK, and shows that $k(x, \xi)$ and $y(x)$ are the unique solutions of them. Some illustrative applications to ordinary differential equations are given.

By extending the results to systems of n integral equations, the author shows that the matrix exponential function $Z(A)=e^A$ satisfies the total differential system

$$\delta Z=\int_0^1 Z((1-\xi)A)\cdot\delta A\cdot Z(\xi A)d\xi, \quad Z(0)=I.$$

Corresponding results are stated for Fredholm integral equations. *F. Smithies* (Cambridge, England).

Dubrovsky, V. M. On a class of integral equations with a variable domain of integration. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 614–616 (1945). [MF 14413]

The author gives conditions under which the equation

$$u(x)=f(x)+\lambda\int_{-\infty}^{x+h}K(x,y)u(y)dy,$$

where $f(x)$ and $K(x, y)$ are given real functions and h and λ are real constants, has a unique solution given by

$$u(x)=\sum_{n=0}^{\infty}u_n(x), \quad u_0(x)=f(x),$$

$$u_n(x)=\lambda\int_{-\infty}^{x+h}K(x,y)u_{n-1}(y)dy.$$

The sufficient conditions stated (apart from measurability) are: $\Phi(x)=\int_{-\infty}^{x}L(y)dy<\infty$ for every real x, where $L(y)=\text{u.b.}\,|K(x,y)|$ for $y-h \leqq x<\infty$; $\lim_{h\to 0}H(h)=0$, where $H(h)=\text{u.b.}\,\{\Phi(x+h)-\Phi(x)\}$ for $-\infty<x<\infty$;

$$I(x)=\int_{-\infty}^{x}L(y)\,|f(y)|\,dy<\infty, \quad -\infty<x<\infty;$$

$\lim_{x\to+\infty}I(x+h)/I(x)=1$; $|\lambda|H(h)e<1$. The author deduces as an immediate corollary that the equation

$$u(x)=f(x)+\lambda\int_{-\infty}^{A}K(x,y)u(y)dy$$

has a unique solution for $x \leqq A$ if $K(x, y)=0$ for $x<A-h$, $x+h<y \leqq A$ and the conditions of the theorem are satisfied merely for $x \leqq A$, $y \leqq A$. *H. R. Pitt* (Belfast).

Trjitzinsky, W. J. Singular Lebesgue-Stieltjes integral equations. Acta Math. **74**, 197–310 (1941). [MF 13770]

The paper contains an adaptation of Carleman's theory of singular integral equations [T. Carleman, Sur les Équa-

tions Intégrales Singulières à Noyau Réel et Symétrique, Uppsala, 1923] to Stieltjes and Lebesgue-Stieltjes integral equations. The Stieltjes integrals used are the integrals with respect to mean additive functions of domains introduced by N. Günther [Sur les Intégrales de Stieltjes et Leurs Applications aux Problèmes de la Physique Mathématique, Leningrad, 1932]. The paper begins with a description of Günther's version of the Stieltjes integral, followed by an examination of the relations between it and the Lebesgue-Stieltjes integral and an extension of known results on weak convergence to functions whose squares are integrable with respect to a given absolutely additive set-function.

The author then considers the integral equation

$$\varphi(x)=\lambda\int_{(D_y)}k(\tau,x)\varphi(y)d\tau+f(x)$$

for the unknown point function $\varphi(x)$, and the related equation

$$\psi(\tau)=\lambda\int_{(D_x)}k(\tau,x)\psi(\omega)d\omega+F(\tau)$$

for the unknown function of domains $\psi(\tau)$. In both equations the integrals are to be taken in the Günther-Stieltjes sense. The kernels are required to be symmetric in a generalised sense and to satisfy certain continuity and finiteness conditions; the equations are supposed to be singular, that is, they do not satisfy the conditions under which Günther's adaptation of the Fredholm theory is valid. As in the Carleman theory, the author approximates to the kernels by sequences of nonsingular kernels and constructs spectral functions (here both point-functions and functions of domains) as limits of sequences of spectral functions for the approximating kernels. He then considers equations involving Lebesgue-Stieltjes integrals, finding spectral representations for the solutions of the nonhomogeneous equation for all nonreal and certain real values of λ. Results analogous to Carleman's are obtained for the homogeneous equation.

The paper concludes with some remarks on the extension of the theory to kernels not satisfying the continuity and finiteness conditions assumed earlier and a discussion of integral equations of the first kind. *F. Smithies.*

Parodi, Maurice-Alexandre. Sur deux équations intégrales. C. R. Acad. Sci. Paris **221**, 485–487 (1945). [MF 14689]

By use of the Laplace transform, formal solutions are obtained of integral equations of the forms

$$\int_0^\infty h(x+t)f(x)dx=g(t),\quad \int_t^\infty h(x-t)f(x)dx=g(t),$$

where $f(x)$ is the unknown function. *H. Pollard.*

Bateman, H. Some integral equations of potential theory. J. Appl. Phys. **17**, 91–102 (1946). [MF 15409]

This paper surveys singular integral equations, generally with Poisson kernels or variants of them, which arise in problems of geophysics. By way of illustration we mention the equation

$$f(x)=\pi^{-1}\int_{-\infty}^{\infty}yF(t)\{(x-t)^2+y^2\}^{-1}dt$$

which, for fixed y, is inverted under suitable conditions by

$$F(x)=f(x-iy)+f(x+iy)-\pi^{-1}\int_{-\infty}^{\infty}yf(t)\{(x-t)^2+y^2\}^{-1}dt.$$

There is considerable emphasis on practical methods of solution. *H. Pollard* (New Haven, Conn.).

Functional Analysis, Ergodic Theory

Dieudonné, Jean. Sur le théorème de Lebesgue-Nikodym. II. Bull. Soc. Math. France **72**, 193–239 (1944). [MF 13228]

This article completes the work started in part I [Ann. of Math. (2) **42**, 547–555 (1941); these Rev. **3**, 50] and establishes necessary and sufficient conditions for the abstract theorem of Lebesgue-Nikodym: if U is a positive linear function on a space E, then each positive linear function L on E may be represented by $L=V+S$, where V is "absolutely continuous" and S is "singular" in the sense that $\inf(U,S)=0$. The "ring of Riesz" for which the theorem is true is shown to be the ring of functions summable with respect to a completely additive measure on a set Ω. Hence the abstract theorem is essentially no more general than the classical one.

A Riesz space is a vector space E with an order relation such that (I) E is a lattice, (II) $x\leqq y$ implies $x+z\leqq y+z$, (III) $x\in E$, $x\geqq 0$, and λ real and nonnegative implies $\lambda x\geqq 0$. In terms of positive linear forms constituting seminorms on E defining a locally convex topology on the space, a completion of the space to $\bar{E}$ is given carrying an extension of order and limit principles involving filters and leading to a coherent Riesz space, that is, one in which each bounded portion has a supremum. A Riesz space is said to be a Riesz ring if in addition the space is a commutative ring with the property $\lambda(xy)=(\lambda x)y$ for each scalar λ and satisfies (IV) x and y in E and $x\geqq 0$ imply $\sup(xy,xz)=x\sup(y,z)$. On the Riesz ring there is supposed given a positive linear form U such that (V_a) $U(x^2)=0$ implies $x=0$, which is weaker than the condition "$U(|x|)=0$ implies $x=0$" of the previous paper. The completion of E in terms of U is called E_U. An element $x\in E_U$ is said to be reasonable (modéré) if there is an element $a\geqq 0$ such that $|x|\leqq a$.

Statements of the two main theorems follow. (1) Being given a Riesz ring E and a positive linear form U on E satisfying (V_a), in order that the theorem of Lebesgue-Nikodym shall hold for E and U it is necessary and sufficient that (VII_a) for each $x\in E_+$ one has $U(x)=\sup U(y,x)$ as y runs over S. (2) The same statement as above with (VII_a) replaced by: there exists an isomorphism ψ of E on a Riesz ring formed of classes of functions summable on a set Ω for a measure μ, such that, for each $x\in E$ and each u_x of class $\psi(x)$, one has $U(x)=\int_\Omega u_x(\xi)d\mu$.

Some notion of the sequence of steps in proving these theorems will be given after stating (VII) and (VII_b). (VII) If X is a linear function relatively bounded on E such that $0\leqq X\leqq U$ then $X(|x|)=\sup|X(yx)|$ for $x\in E$, the supremum being taken over all y such that $|y|\in S$, where S is the set of all $z\in E$ such that $zx\leqq x$ whenever $x\in E_+$. (VII_b) For all $x\in E_+$, one has $U(x)=\sup U_y(x)$, the supremum being taken over all y such that $y\in T$, where T is the set of those elements of E_U such that $yx\leqq x$ whenever $x\in E_+$ and U_y is the function defined for each $x\in E$ by $U_y(x)=U(yx)$. Clearly (VII) implies (VII_a) implies (VII_b). The author establishes that the validity of the first theorem above with (VII_a) replaced by (VII_b) implies the isomorphism of E with a ring of classes of summable functions, and finally establishes that for a ring of this nature the condition (VII) is true, thus establishing the two theorems stated above and the equivalence of (VII), (VII_a) and (VII_b).

In the second part of the paper the condition (VII_a) on E and U is relinquished but (V_a) is kept. It is found that, if E is a Riesz ring and U is a positive linear form satisfying (V_a) on E, then (1) there exists one and only one element e

of E_U such that $ex=x$ for each $x\varepsilon E$ and (2) U_e is a positive linear form on E satisfying (V_s) and (VII_s), and $V=U-U_e$ is a positive linear form on E such that $V(x^2)=0$ for each $x\varepsilon E$. A substitute for the Lebesgue-Nikodym theorem is then found as follows: each positive linear form X on E may be written in one and only one way as $X=U_y+Y+Z$, where $y\varepsilon E_U^+$, Y belongs to the complete family of V and is thus such that $Y(x^2)=0$ for each $x\varepsilon E$ and Z is disjoint from U. *J. F. Randolph* (Oberlin, Ohio).

Nakano, Hidegorô. Topologische Masse. Proc. Phys.-Math. Soc. Japan (3) **25**, 279–334 (1943). [MF 15057]

The author gives an exhaustive treatment of measure and integration in topological spaces, and, more generally, in topological Boolean algebras. The general idea is to define measure on a class of sets which is a neighborhood basis for the topology and then to extend the measure to all Borel sets. The details are too technical to be outlined here, but the following illustrates the point of view. The topological assumptions are so broad that the theorem that an additive measure function $\mu(K)$ defined on a field G of sets can be extended to be completely additive on the Borel field determined by G if $K_n \downarrow 0$ implies that $\mu(K_n)\to 0$ is derived by introducing a topology in which the sets of G constitute a neighborhood basis. (These sets are then both open and closed.) *J. L. Doob* (Urbana, Ill.).

Kawada, Yukiyosi. Über die Erweiterung der maximalen Ideale in normierten Ringen. Proc. Imp. Acad. Tokyo **19**, 267–268 (1943). [MF 14821]

If R_1 is a normed ring (commutative) and R_2 is one of its subrings, the ideal extension problem is the following. Under what conditions on R_1 and R_2 is every maximal ideal of R_2 a part of some maximal ideal of R_1? Conditions given here which guarantee this are: R_1 contains with every element x an $\bar{x}$ defined by the condition that the function $\bar{x}(M_1)$ is the complex conjugate of $x(M_1)$, where M_1 represents an arbitrary maximal ideal of R_1. Also, if $x\varepsilon R_2$ then $\bar{x}\varepsilon R_2$. *E. R. Lorch* (New York, N. Y.).

Kawada, Yukiyosi. Über den Operatorenring Banachscher Räume. Proc. Imp. Acad. Tokyo **19**, 616–621 (1943). [MF 14859]

The author observes that the minimal left ideals in the ring of bounded linear operators on a Banach space X are equivalent as Banach spaces to the original X and that the minimal right ideals bear the same relationship to the conjugate space $\bar{X}$. He then applies this observation to the further study of the relationship between a Banach space and its ring of bounded linear operators. In addition to giving alternative proofs of known theorems of Eidelheit, Kakutani and the reviewer he shows that if the "adjoint" of the ring of a Banach space is isomorphic as a ring to the ring of some other Banach space then both spaces are reflexive and one is isomorphic to the conjugate of the other. He gives a necessary and sufficient condition that a normed ring be the ring of all bounded linear operators on a Hilbert space and a sufficient condition in terms of the ring that a Banach space be the direct sum of a Banach space and its conjugate. Furthermore, he states without formal proof that a Banach space is isomorphic to its conjugate if and only if its ring admits an operation $A\to A^*$ such that $A^{**}=A$, $(AB)^*=B^*A^*$ and $(\alpha A+\beta B)^*=\bar{\alpha}A^*+\bar{\beta}B^*$, where A and B are elements of the ring and α and β are scalars. [The reviewer does not see how the methods of the author may be used to establish the "only if" part of this theorem without assuming that the Banach space is reflexive. As far as the reviewer has been able to determine it is not known whether or not a Banach space which is isomorphic to its second conjugate is necessarily reflexive.] *G. W. Mackey* (Cambridge, Mass.).

Izumi, Shin-ichi. Notes on Banach space. V. Compactness of function spaces. Proc. Imp. Acad. Tokyo **19**, 99–101 (1943). [MF 14796]

[Note IV, by Nakamura and Sunouchi [same Proc. **18**, 333–335 (1942)] is not related to this note.] The author proves a general theorem on compactness of subsets of linear spaces of type F [see Banach, Théorie des Opérations Linéaires, Warsaw, 1932, p. 35], and applies the result to give necessary and sufficient conditions for compactness of subsets of the set of measurable functions defined on the closed unit interval. The metric used is given by the norm

$$\|f\| = \int_0^1 \frac{|f(t)|}{1+|f(t)|}\,dt.$$

E. Hewitt (Princeton, N. J.).

Izumi, Shin-ichi, and Sunouchi, Gen-ichirô. Notes on Banach space. VI. Abstract integrals and linear operations. Proc. Imp. Acad. Tokyo **19**, 169–173 (1943). [MF 14808]

[Cf. the preceding review.] Let X be a Banach space of functions $\phi=\phi(t)$ on a range T, where the scalar $\phi(t)$ for every fixed t is linear and continuous in ϕ. The general linear operation from an arbitrary Banach space Y to X is of the form $Uy=\bar{y}(\cdot)y$, where $\bar{y}(\cdot)=\bar{y}(t)$ is a function on T to Y with the property that for every $y\varepsilon Y$ the scalar function $\bar{y}(t)y$ is a representation of a point in X. Except for the explicit evaluation of the norm the theorem contains most of the well-known representation theorems for linear operations. Application is also made to the representation of a type of linear operation considered by Kantorovitch and Vulich on a partially ordered Banach space. *N. Dunford* (New Haven, Conn.).

Sunouchi, Gen-ichirô. Notes on Banach space. VII. Compactness of function spaces. Proc. Imp. Acad. Tokyo **19**, 424–425 (1943). [MF 14836]

[Cf. the two preceding reviews.] A remark on Banach spaces of continuous functions, the meaning of which is not clear to the reviewer. *E. Hewitt* (Princeton, N. J.).

Nakamura, Masahiro, and Kakutani, Shizuo. Banach limits and the Čech compactification of a countable discrete set. Proc. Imp. Acad. Tokyo **19**, 224–229 (1943). [MF 14814]

Let $\bar{\Omega}$ be the Čech compactification of a countable discrete set Ω. It is clear that the Banach space (m) of all bounded real sequences is equivalent to the space $C(\bar{\Omega})$ of all real continuous functions defined on $\bar{\Omega}$. Hence linear functionals on (m) may be represented by Lebesgue integrals over $\bar{\Omega}$. The Banach limits correspond precisely to the measures which vanish identically on Ω. A short proof is given for the result of G. Fichtenholz and L. Kantorovitch which states that the space conjugate to (m) and the family of all Banach limits have the same cardinal number 2^c. It is shown that there exists a family of mutually disjoint nonempty open and closed subsets of $\bar{\Omega}-\Omega$ with the cardinal number c. *N. Dunford* (New Haven, Conn.).

Hanai, Sitiro. On the double sequences of linear transformations on normed linear separable spaces. Jap. J. Math. 19, 27–31 (1944). [MF 14992]

Let $\{U_n^m\}$ be a double sequence of continuous mappings of a space X into a space Y, such that $\lim_{m\to\infty}(\lim_{n\to\infty} U_n^m(x))$ exists for all $x\in X$. There will sometimes exist sequences $\{m_k\}$ and $\{n_k\}$ such that

$$\lim_{k\to\infty} U_{n_k}^{m_k}(x)=\lim_{m\to\infty}(\lim_{n\to\infty}(U_n^m(x))$$

for all $x\in X$. The author discusses this question for separable metric spaces and certain normed linear spaces.

E. Hewitt (Princeton, N. J.).

Godement, Roger. Sur une généralisation d'un théorème de Stone. C. R. Acad. Sci. Paris 218, 901–903 (1944). [MF 15238]

This paper sketches the spectral analysis of a weakly continuous unitary representation in Hilbert space (not necessarily separable) of a locally compact Abelian group, arriving at results similar to those obtained by Neumark [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 237–244 (1943); these Rev. 5, 272] and Ambrose [Duke Math. J. 11, 589–595 (1944); these Rev. 6, 131].

M. H. Stone (Cambridge, Mass.).

Julia, Gaston. Sur les projections des systèmes orthonormaux de l'espace hilbertien. C. R. Acad. Sci. Paris 218, 892–895 (1944). [MF 15236]

The author considers a linear closed manifold h of infinite dimensionality in a Hilbert space H and orthonormal bases ϵ_k and e_k of H and h, respectively. Let $A_k=P_h\epsilon_k$ $(k=1, 2, \cdots)$ be the projection of ϵ_k on h and let $Ax\equiv\sum x_kA_k$, where $x=\sum x_ke_k$. He then proposes to determine the character of A in terms of an operator defined over h. If J is an isometric operator over H and if $w=J(h)$, then it can be seen that $A=J^*P_w$, where P_w is the projection on w. He now defines J_1 to be J^* on w and null on $h\ominus w$, the orthogonal complement of w in h. It is then clear that $A=J_1$ on h, that J_1 is isometric on w, $h=J_1(w)$, $J_1(h\ominus w)=0$, $J_1J_1^*=1$, $J_1^*J_1=P_w$, and that $h\ominus w$ has the same dimensionality as $H\ominus h$; moreover the attained upper bound of A^*A is 1. Conversely, if w is an arbitrary linear closed manifold in h such that $h\ominus w$ has the dimensionality of $H\ominus h$, one can find an operator J_1 for which $A_k\equiv J_1P_we_k$ are the projections of ϵ_k on h. The author then proceeds to obtain analogous formulas characterizing a sequence of elements A_k which are the projections of an incomplete orthonormal sequence E_k.

H. H. Goldstine (Princeton, N. J.).

Julia, Gaston. Sur la représentation analytique des opérateurs bornés ou fermés de l'espace hilbertien. C. R. Acad. Sci. Paris 219, 225–227 (1944). [MF 15253]

Let h be a Hilbert space and let it be embedded in the Hilbert space $H=(h, h)$. Then the author shows that each bounded operator A on h is expressible as M_AP_hU, where U is a bounded operator on H and is isometric on h, P_h is the projection operator on h and M_A is the bound of A. If A is a closed operator, he represents it in parametric form as $x=Bz$, $Ax=Cz$, where B and C are bounded operators on h, defined as follows. Let V_A be the locus of points $z=[x, Ax]$, x in h. Then V_A is closed and has an orthonormal base E_k, which can be decomposed into $e_k=P_hE_k$ in h and $e_k'=P_h'E_k$ in $H\ominus h$. If $\sum|z_k|^2$ converges, so do $x=\sum z_ke_k$ and $[0, Ax]=\sum z_ke_k'$. It is, moreover, shown that every x in the domain of existence of A is thus expressible and $Bz\equiv\sum z_ke_k$. If U is unitary, transforming h into h', then $Ax=\sum z_kUe_k'$.

H. H. Goldstine (Princeton, N. J.).

Van der Lijn, G. Sur les équations intégrales à noyau symétrique. Bull. Soc. Roy. Sci. Liége 10, 218–223 (1941). [MF 13061]

The author proves (i) that, if $U(x, y)$ is a continuous symmetric bilinear operator from Hilbert space into a normed linear space, then

$$\sup_{\|x\|,\,\|y\|\leq 1}\|U(x, y)\|=\sup_{\|x\|\leq 1}\|U(x, x)\|;$$

(ii) that, if $\varphi(x)$ is a bounded linear symmetric operator in a complex Hilbert space, not identically zero, then, given $\epsilon(>0)$, there exist an element x of norm 1 and a complex number λ such that $\|x-\lambda\varphi(x)\|<\epsilon$.

F. Smithies.

Hanai, Sitiro. Completely continuous transformations in Hilbert spaces. Proc. Imp. Acad. Tokyo 21, 79–82 (1945). [MF 14951]

Let $\mathfrak{T}$ denote the Banach space of all completely continuous transformations defined in a separable Hilbert space $\mathfrak{H}$, with the bound of a transformation as its norm. The author proves (i) that there is a linearly independent double sequence $\{T_{ij}\}$ of elements of unit norm and a double sequence $\{a_{ij}(T)\}$ of bounded linear functionals such that for any $T\in\mathfrak{T}$

$$T=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}(T)T_{ij},$$

(ii) that $\mathfrak{T}$ is a separable space of type A in the sense of I. Maddaus [Bull. Amer. Math. Soc. 44, 279–282 (1938)]. The proofs as given are not quite complete, but the missing details can be supplied without difficulty.

F. Smithies.

Munroe, M. Evans. A note on weak differentiability of Pettis integrals. Bull. Amer. Math. Soc. 52, 167–174 (1946). [MF 15457]

Phillips [Trans. Amer. Math. Soc. 47, 114–145 (1940); these Rev. 2, 103] gave a function on the unit interval with values in Hilbert space whose Pettis integral is not almost everywhere weakly differentiable. The writer constructs such a function with values in the space C of continuous functionals on the unit interval. Using this special example he is able to construct such a function with values in the space of continuous functionals on any compact metric space E containing uncountably many points.

M. M. Day (Urbana, Ill.).

Zorn, Max A. Derivatives and Fréchet differentials. Bull. Amer. Math. Soc. 52, 133–137 (1946). [MF 15453]

Let X and Y be complex Banach spaces, and $f(x)$ a function defined on an open subset S in X, with values in Y. The Gâteaux differential $\delta f(x; h)$ is defined as the derivative of $f(x+th)$ with respect to the complex variable t at $t=0$. The following theorem is proved. If the Gâteaux differential exists and is a continuous linear function of h for each x in S, then it is a Fréchet differential. This occurs as a corollary of the more general theorem: if the Gâteaux differential exists for each x of S, and if it is a continuous linear function of h for each x in a nonvoid, finitely open subset D of S, it satisfies the condition $f(x+h)-f(x)-\delta f(x; h)=o(\|h\|)$ at each point of D. A set D in X is called finitely open if for arbitrary choices of x in D, of a positive integer n, and $h_1, \cdots, h_n$ in X, the $n$uples $(t_1, \cdots, t_n)$ for which $x+t_1h_1+\cdots+t_nh_n$ is in D form an open set in the number space of the variables $t_1, \cdots, t_n$. The proof is arranged around the use of higher order Gâteaux differentials and power series expansions, concerning which results are borrowed from another paper of the author [Ann. of Math. (2) 46, 585–

593 (1945); these Rev. 7, 251]. When the Gâteaux differential at x is a bounded linear function of h, it is regarded as defining a function of x with values in the space $[X, Y]$ of bounded linear transformations of X into Y. This function is called the derivative of $f(x)$. *A. E. Taylor.*

Zorn, Max A. Gâteaux differentiability and essential boundedness. Duke Math. J. **12**, 579–583 (1945). [MF 15503]

Consider a connected open set D in a complex Banach space X, and a function $f(x)$ defined on D, the values of the function lying in a second complex Banach space Y. Suppose that $f(x)$ possesses a Gâteaux differential [see the definition in the preceding review] at each point of D. The author points out three known theorems, each of which adds some hypothesis about the behavior of $f(x)$ on D, and then draws the conclusion that $f(x)$ possesses a Fréchet differential at each point of D. The three additional hypotheses are, respectively: (1) continuity of $f(x)$ on D [A. E. Taylor, Ann. Scuola Norm. Super. Pisa (2) **6**, 277–292 (1937)]; (2) local boundedness of $f(x)$ on D [E. Hille, Amer. Math. Soc. Colloquium Lectures, 1944]; (3) continuity of $f(x)$ on $D-P$, where P is a set of the first category [Zorn, Ann. of Math. (2) **46**, 585–593 (1945); these Rev. **7**, 251]. The present paper gives a theorem which contains all three of the theorems referred to. The statement of the theorem depends upon a notion introduced by Mazur and Orlicz [Studia Math. **5**, 50–68, 179–189 (1934)]. A set P in X is called an M-O set if X is not the sum of a countable number of homothetic images $\alpha_n P+a_n$ of P (α_n complex, a_n in X). The theorem is that, if $f(x)$ possesses a Gâteaux differential at each point of D, and if there exists a neighborhood U of some point in D and an M-O set P such that f is bounded in $U-P$, then $f(x)$ is Fréchet differentiable on D. The proof depends upon results developed in connection with the last two of the three theorems cited above, in combination with the following lemma. Let D be a set in X such that, for all pairs x, h in X, the set of complex numbers t for which $x+th$ is in D is open. Let $f(x)$ be defined on D and suppose that $f(x+th)$ is continuous in t for every pair x, h. Finally, suppose that there exists an M-O set P such that $f(x)$ is bounded on $D-P$. Then $f(x)$ is bounded on D. *A. E. Taylor* (Los Angeles, Calif.).

Michal, A. D. First order differentials of functions with arguments and values in topological Abelian groups. Revista Ci., Lima **47**, 389–422 (1945). [MF 15428]

By a topological Abelian group the author means an additively written commutative group on which is imposed a topology of the type of Fréchet's classes (H) [cf. Sierpiński, Introduction to General Topology, University of Toronto Press, 1934, p. 33]. In addition to requiring that the group operations are continuous, the author requires that, given any element y and any neighborhood U of the zero element, there exists a positive integer n such that $y=nx$ for some x in U. The latter restriction is used only in proving that differentials as defined are unique.

Let $f(x)$ be a function with argument in T_1 and values in T_2, where T_1 and T_2 are topological Abelian groups. If $f(x_0; \delta x)$ is to be the differential of $f(x)$ at $x=x_0$, with increment δx, it must be linear (additive and continuous) in δx, and in some suitable sense it must be a first order approximation to $f(x_0+\delta x)-f(x_0)$. The paper contains eight different definitions of a first order differential, corresponding to different ways of rendering precise this requirement of first order approximation. An additional eight types are obtained by dropping the requirement that the differential is continuous in δx. The least restrictive of the first eight definitions is that of what the author calls the M_1 differential. If $f(x_0; \delta x)$ is linear in δx, it is called the M_1 differential of $f(x)$ at $x=x_0$ if there exists a function $\epsilon(x_0, x_1, x_2)$ with arguments in T_1 and values in T_2 such that the following four conditions are satisfied: (1) $\epsilon(x_0, 0, x)\equiv 0$; (2) $\epsilon(x_0, x_1, nx_2) = n\epsilon(x_0, x_1, x_2)$ for all positive integers n, for all x_1 in some neighborhood of 0, and for all x_2; (3) there exists a neighborhood W of 0 in T_1 with the property that to each neighborhood V of 0 in T_2 corresponds a neighborhood U of 0 in T_1 such that $\epsilon(x_0, x_1, x_2)$ is in V if x_1 is in U and x_2 is in W; (4) there exists a neighborhood N of 0 in T_1 such that, if δx is in N, $f(x_0+\delta x)-f(x_0)-f(x_0; \delta x)=\epsilon(x_0, \delta x, \delta x)$.

If the M_1 differential exists, it is unique, and $f(x)$ is continuous at x_0. The theorem about the differential of a function of a function holds in the same form as for Fréchet differentials. Various modifications of the M_1 differential are defined by placing additional requirements on ϵ as a function of x_1 and x_2, either together or singly. If T_1 and T_2 are real Banach spaces, and if in addition $\epsilon(x_0, x_1, tx_2) = t\epsilon(x_0, x_1, x_2)$ for all positive values of t, then the M_1 differential and its modifications coincide with the Fréchet differential. *A. E. Taylor* (Los Angeles, Calif.).

★Fantappié, L. Teoría de los Funcionales Analíticos y sus Aplicaciones. [Theory of Analytic Functionals and its Applications]. Consejo Superior de Investigaciones Científicas, Barcelona, 1943. 174 pp. (Spanish)

This memoir presents a study of the functionals on a space of analytic functions, topologized by a Hausdorff topology in which neighborhoods are set up in terms of the maximum difference of functions on regions. The functionals $F(y(t))$ are restricted to those which will retain analyticity relative to a parameter in $y(t)$, that is, $F(y(t, \alpha))$ is analytic in α if $y(t, \alpha)$ depends analytically on α and is subject to a condition which depends on the domain of F. Linear functionals are studied by considering the indicatrix functions $u(\alpha)=F(1/(t-\alpha))$ and $w(\alpha)=F(1/(1-t\alpha))$. A line integral representation for F in terms of these functions is obtained. This representation is the one indicated by the Cauchy integral formula and the linearity of F. The notion of a linear operator is considered in the form of a "mixed" functional $N=F(y(t), z)$ which is numerically valued. If N is a linear operator which takes $y(t)$ into $f(z)$, then $F(y(t), z)$ is the value of f at z. A "normal operator" is defined as one which has the powers t^n as characteristic functions. For normal operators, the spectral function

$$w(z, \alpha)=F(1/(1-\alpha t), z)$$

is shown to depend upon the product $z\alpha$ only, that is, $w(z, \alpha)=k(z\alpha)$. The set of operators which commute with differentiation are characterized by the property that $u(z, \alpha)=F(1/(\alpha-t), z)$ depends only on $z-\alpha$. Bilinear functionals are also introduced. The next step is the consideration of nonlinear analytic functionals in general. For these the author establishes the usual concepts of the differential calculus such as the derivative, with the customary properties and rules and the series expansion of such a function. The corresponding questions for functions of more than one variable are also considered but the results are not as complete. However, these results contain a Cauchy integral formula for the many-variable case, which permits an operational calculus of a somewhat restricted sort. As an application the Cauchy theory of partial differential equations is considered. In certain cases, the Cauchy problem is solved.

The geometrical significance in abstract spaces of such notions as characteristic strips and singular solutions is given. A similar discussion of the notions of hyperbolic, parabolic and elliptic partial differential equations which pertain to the case of two independent variables also appears.
F. J. Murray (New York, N. Y.).

Fan, Ky. Two mean theorems in Hilbert space. Proc. Nat. Acad. Sci. U. S. A. **31**, 417–421 (1945). [MF 14533]
Let $x_1, x_2, \cdots$ be elements of a Hilbert space. The author proves the existence of the strong limit $\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} x_i$ under the following assumptions. There exist two constants c_1 and c_2 such that for all m, n

(1) $\|x_{n+1}+\cdots+x_{n+m}\|^2-\|x_1+\cdots+x_m\|^2<c_1 m,$

(2) $\|x_1+\cdots+x_{n+2}\|^2-2\|x_1+\cdots+x_{n+1}\|^2+\|x_1+\cdots+x_n\|^2>c_2.$

The result holds if (2) is replaced by the existence of

$$\lim_{n\to\infty} n^{-1}\{\|x_1+\cdots+x_{n+1}\|^2-\|x_1+\cdots+x_n\|^2\}.$$

E. R. Lorch (New York, N. Y.).

Fan, Ky. Remarques sur un théorème de M. Khintchine. Bull. Sci. Math. (2) **69**, 81–92 (1945). [MF 15416]
Proofs of theorems stated in the paper reviewed above and in C. R. Acad. Sci. Paris **220**, 102–104 (1945); these Rev. **7**, 128. *J. L. Doob* (Urbana, Ill.).

Nakamura, Masahiro. Note on Banach spaces. II. An ergodic theorem for Abelian semi-groups. Proc. Imp. Acad. Tokyo **18**, 131 (1942). [MF 14745]
[Note I, by Izumi [same Proc. **18**, 127–130 (1942)], is not related to this note.] A proof based upon the Riesz-Yosida form of the mean ergodic theorem is given for the following theorem of G. Birkhoff and L. Alaoglu: every reflexive Banach space is ergodic under a uniformly bounded Abelian semigroup of linear operators. *N. Dunford.*

Izumi, Shin-ichi. A remark on ergodic theorems. Proc. Imp. Acad. Tokyo **19**, 102–104 (1943). [MF 14797]
The author sketches the proofs of Birkhoff's ergodic theorem, Wiener's dominated ergodic theorem and some of their generalizations to vector lattices. In addition to a theorem of Kantorovitch on the convergence of sequences of transformations from one regular vector lattice to another, quite heavy use is made also of the customary machinery of proofs of ergodic theorems. *P. R. Halmos.*

Riesz, F. Rectification au travail "Sur la théorie ergodique des espaces abstraits." Acta Litt. Sci. Szeged **10**, 141 (1941). [MF 15841]
The paper appeared in the same Acta **10**, 1–20 (1941); these Rev. **2**, 318.

Theory of Probability

Kawada, Yukiyosi. Über eine verbandstheoretische Begründung der Wahrscheinlichkeitsrechnung. Jap. J. Math. **18**, 887–972 (1943). [MF 14988]
The author gives a complete treatment of the foundations and basic theorems of probability from the lattice theoretic point of view. In the set theory treatment, an elementary event is a point of an abstract space Ω; a completely additive set function (probability measure) is defined on certain sets of Ω; a chance variable is a function f, measurable with respect to probability measure, and the expectation of f is its integral over Ω. The author drops the concept of an elementary event and takes as basic that of a chance variable. The following is a rough outline of his approach. Chance variables x are postulated as objects which can be added to each other and multiplied by real numbers, the usual rules of associativity, etc. being satisfied, and $E\{x\}$, the expectation of x, is a functional satisfying rules of linearity, etc. An order relation is postulated, such that if $x_1 \geqq x_2$, $E(x_1) \geqq E(x_2)$; and if $x_1 \geqq x_2 \geqq \cdots$, $\wedge_n x_n = 0$, then $E(x_n) \to 0$. The relation $x_1 \geqq x_2$ is interpreted in the set theory approach as the corresponding functional inequality, satisfied everywhere on Ω. A probability field is defined as a linear space of chance variables which is a σ-complete vector lattice. This can be interpreted in the set theory approach as the set of bounded chance variables or the set of integrable chance variables, etc. The theorems of probability now become theorems on probability fields. The analysis is taken through the strong law of large numbers, which becomes (omitting the details of the hypotheses) the statement that, if $x_1, x_2, \cdots$ are equidistributed mutually independent chance variables in a probability field, there is a chance variable w in the field, with $w \geqq 0$ and $E\{w\} = 0$, such that if $f(x)$ is any functional (of a certain delimited class) defined on the field, for which $f(w) = 0$, it follows that $\lim_{n\to\infty} n^{-1}\{f(x_1)+\cdots+f(x_n)\} = E\{x_1\}$.
J. L. Doob (Urbana, Ill.).

Féraud, Lucien. Les notions de loi et d'hypothèse probabilistes. Arch. Sci. Phys. Nat., Geneva **27**, 191–208 (1945). [MF 15414]

Kaplansky, Irving, and Riordan, John. Multiple matching and runs by the symbolic method. Ann. Math. Statistics **16**, 272–277 (1945). [MF 13917]
This paper contains a systematic treatment of the symbolic method as applied to the derivation of formulas arising in the theory of probability, in particular, in the multiple matching problem (matching of cards drawn at random from several decks). The formulas are expressed symbolically in terms of the displacement operator E defined by the relation $E\varphi_x = \varphi_{x+1}$. The method is especially suited to the computation of factorial moments of probability distributions. The authors give several examples of such computations. *A. H. Copeland* (Ann Arbor, Mich.).

Price, G. B. Distributions derived from the multinomial expansion. Amer. Math. Monthly **53**, 59–74 (1946). [MF 15462]

Erdös, P. On a lemma of Littlewood and Offord. Bull. Amer. Math. Soc. **51**, 898–902 (1945). [MF 14456]
Improving a result of Littlewood and Offord [Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 277–286 (1943); these Rev. **5**, 179], the author proves that, if $x_1, \cdots, x_n$ are complex numbers such that $|x_i| \geqq 1$ $(i = 1, \cdots, n)$, then the number of sums $\sum_1^n \pm x_i$ which fall within a circle of radius r is not greater than $cr2^n n^{-\frac{1}{2}}$ (c, a constant). A number of related results are also included. *M. Kac* (Ithaca, N. Y.).

Goudsmit, S. Random distribution of lines in a plane. Rev. Modern Phys. **17**, 321–322 (1945). [MF 13697]
The author considers a plane covered with straight lines which are distributed at random in position and direction. The mean area $\langle\sigma\rangle_{Av}$ and the mean square area $\langle\sigma^2\rangle_{Av}$ of the fragments thus cut out are calculated in terms of the mean length of the sections into which the lines are cut. If the

mean area is normalized by $\langle\sigma\rangle_{Av}=1$, the mean length of the sections is $\frac{1}{2}\pi^{\frac{1}{2}}$. The mean square area is found to be $\langle\sigma^2\rangle_{Av}/\langle\sigma\rangle_{Av}^2=\frac{1}{2}\pi^2$. This result is obtained by expressing the probability P_2 for two arbitrarily chosen points to lie in the same fragment by means of $\langle\sigma^2\rangle_{Av}$ and $\langle\sigma\rangle_{Av}$ and then calculating P_2 as the probability that the line connecting two arbitrarily chosen points is not intersected by another line in the interval between the points.

For the case that the straight lines are random in position, but are specified to be parallel to either one of two orthogonal axes, the probability density of the fragment areas is given. In this case $\langle\sigma^k\rangle_{Av}=(k!)^2$, provided that the average length between intersections is normalized to unity.

A. J. F. Siegert (Syracuse, N. Y.).

Finsler, P. Über die Wahrscheinlichkeit seltener Erscheinungen. Experientia 1, 56–57 (1945). [MF 15385]

Consider a sequence of random events subject to a Poisson distribution with parameter δ. Let $w(\delta; t, T)$ denote the probability that a time interval of length T contains at least one subinterval of length t within which no event has occurred. The author states without proof that

$$w(\delta; t, T) = \sum(-1)^{n-1}\delta^{n-1}(T-nt)^{n-1}(1+n^{-1}\delta(T-nt))\,e^{-n\delta t}/(n-1)!,$$

the summation extending over $1\leqq n\leqq[t/T]$. *W. Feller.*

van Kreveld, A. Dispersion of the distribution in space of particles of different classes. Physica 8, 1045–1058 (1941).

Particles (fat globules in milk) are classified into classes according to size. The number of particles of one class in a certain volume is a random variable. In the simplest case these random variables are mutually independent and Poisson-distributed. The author introduces measures of dispersion as ratios of actually occurring variances to the variances corresponding to the simple Poisson scheme. The measures of dispersion are computed in some typical cases with only two classes A and B present. Typical assumptions are that the probability of finding a particle of one class depends only on the position of particles of the other class, or alternatively that only particles B have an influence on particles A. *W. Feller* (Ithaca, N. Y.).

Berg, W. F. Aggregates in one- and two-dimensional random distributions. (Developability of silver specks of known dimensions and the size of photographic sensitivity specks.) Philos. Mag. (7) 36, 337–346 (1945). [MF 15355]

In a sequence of random events obeying the Poisson law a "burst" is said to occur if an interval of length δ contains exactly k events. In first approximation the author assumes that the number of bursts also has a Poisson distribution. The parameter of the latter can then easily be deduced from the parameter of the given Poisson distribution. Analogous considerations apply naturally to random points in one and two dimensions if the length and area, respectively, are introduced as operational time. Instead of "burst" the term "aggregate" is used for random points. The bulk of the paper is devoted to applications. *W. Feller.*

Silberstein, Ludwik. The probable number of aggregates in random distributions of points. Philos. Mag. (7) 36, 319–336 (1945). [MF 15354]

Suppose that n points of the interval $(0, 1)$ are picked independently and at random so that the coordinate of each point is a random variable with a uniform distribution. If k of the points lie within an interval of length δ they are said to form an aggregate. Using difference equations the author determines approximations to the expected number of aggregates corresponding to various values of n and k. The numerical results are compared with the analogous results for Poisson distributions [cf. the preceding review]. The considerations are generalized to two dimensions.

W. Feller (Ithaca, N. Y.).

Hadwiger, H. Über Verteilungsgesetze vom Poissonschen Typus. Mitt. Verein. Schweiz. Versich.-Math. 45, 257–277 (1945). [MF 15421]

A distribution function depending on the variable x and the continuous parameter $t\geqq 0$ is said to be of the Poisson type if it is a step function with possible jumps only at $x=0, 1, 2, \cdots$ and if it satisfies the Smoluchovski equation $\varphi_n(s+t)=\sum_0^n\varphi_k(s)\varphi_{n-k}(t)$, where $\varphi_n(t)$ denotes the magnitude of the jump at $x=n$; moreover, it is supposed that $\sum n\varphi_n(t)=t$. It is clear that a distribution function is of the Poisson type if and only if its generating function $\sum\varphi_n(t)z^n$ is of the form $\exp(tf(z))$, where $f(z)$ is an analytic function regular for $|z|<1$ with $f(1)=0$, $f'(1)=1$ and $f^{(n)}(0)\geqq 0$ if $n\geqq 1$. It is shown that all solutions share certain qualitative properties with the familiar Poisson distribution. Several examples are given. *W. Feller* (Ithaca, N. Y.).

Hadwiger, H. Die Erfahrungsnachwirkung bei Wahrscheinlichkeiten. Experientia 1, 87–89 (1945). [MF 15387]

The author studies variable probabilities of the form $p(t)=a(a+B(t))^{-1}$, where a and b are constants and

$$B(t)=b-c\int_0^t(1-p(x))dx.$$

One could interpret $p(t)$ by saying that every time an unfavorable event takes place $B(t)$ is reduced by an amount proportional to the expected number of unfavorable events during the time $(0, t)$. It is shown that $p(t)$ satisfies a differential equation of the form $p'=k(p^2-p^3)$, where k is a constant. *W. Feller* (Ithaca, N. Y.).

Risser, René. Étude spéciale du type de tirages de boules d'une urne, renfermant des boules de deux couleurs, dans l'hypothèse de non remise des boules après tirage. C. R. Acad. Sci. Paris 219, 541–542 (1944). [MF 15286]

Risser, René. Sur les courbes de distribution statistique. C. R. Acad. Sci. Paris 219, 505–507 (1944). [MF 15282]

Kac, M. A remark on independence of linear and quadratic forms involving independent Gaussian variables. Ann. Math. Statistics 16, 400–401 (1945). [MF 15476]

The following theorem is proved. If $X_1, \cdots, X_n$ are identically distributed independent Gaussian random variables, each having mean 0, then a necessary and sufficient condition for $\sum_{j,k=1}^n a_{jk}X_jX_k$ and $\sum_{j=1}^n\alpha_jX_j$ to be independent is that $A\alpha=0$, where A is the matrix of the quadratic form and α is the vector $(\alpha_1, \cdots, \alpha_n)$. The most important application is probably to the case $\sum_{j,k=1}^n a_{jk}X_jX_k=\sum_{j=1}^n(X_j-\bar{X})^2$, $\sum_{j=1}^n\alpha_jX_j=\bar{X}=\sum_{j=1}^nX_j/n$. *J. L. Doob* (Urbana, Ill.).

Masuyama, Motosaburô. The Bienaymé-Tchebycheff inequality for Hermitic tensors. Proc. Phys.-Math. Soc. Japan (3) 24, 409–411 (1942). [MF 15029]

If A is a chance variable with expectation M the Bienaymé-Tchebycheff inequality states that

$$(1) \qquad \text{prob}\,\{A-\lambda M\geqq 0\}\leqq 1/\lambda.$$

In the present paper (1) is proved for a chance Hermitian positive definite or semidefinite tensor. If B is a Hermitian tensor, $B \geqq 0$ is interpreted to mean that B is positive definite or semidefinite. A chance tensor is one which takes on various tensor values with preassigned probabilities. If x is a random bivector, the author applies his theorem to $A=(x-\bar{x})(x-\bar{x})^*$; M becomes the variance tensor, and the left side of (1) reduces to the probability that the endpoint of $(x-\bar{x})/\lambda^{\frac{1}{2}}$ lies outside the ellipsoid $xM^{-1}x^*=1$.

J. L. Doob (Urbana, Ill.).

Goddard, L. S. The accumulation of chance effects and the Gaussian frequency distribution. Philos. Mag. (7) **36**, 428–433 (1945). [MF 15480]

Let $f_n(x)$ be the n-fold convolution of the rectangular distribution with itself. Uspensky [Introduction to Mathematical Probability, McGraw-Hill, New York, 1937, p. 278] derives an explicit formula for $f_n(x)$ by the use of characteristic functions, but attributes the formula itself to Laplace. L. Silberstein [Philos. Mag. (7) **35**, 395–404 (1944); these Rev. **6**, 88] computed the first few $f_n(x)$, introduced the Fourier transform and proved convergence to the Gaussian distribution. Haldane [Philos. Mag. (7) **36**, 184–185 (1945); these Rev. **7**, 18] then determined the accuracy of the approximation of $f_n(x)$ to the Gaussian. The same problem is treated by the present author who, however, observes that an explicit formula for $f_n(x)$ has been derived by K. Fränz [Elektr. Nachr. Techn. **17**, 215–230 (1940)]. The standard asymptotic expansions connected with the central limit theorem are ignored in all these papers [see H. Cramér, Random Variables and Probability Distributions, Cambridge University Press, 1937, chapter 7].

W. Feller (Ithaca, N. Y.).

Grimsey, A. H. R. On the accumulation of chance effects and the Gaussian frequency distribution. Philos. Mag. (7) **36**, 294–295 (1945). [MF 15750]

The author points out the integrals occurring in the paper reviewed above are evaluated in "A Treatise on the Integral Calculus" by J. Edwards [Macmillan, London, 1921–1922].

W. Feller (Ithaca, N. Y.).

Franckx, E. L'évolution des collectivités. Mitt. Verein. Schweiz. Versich.-Math. **45**, 279–288 (1945). [MF 15422]

This paper is concerned with the much discussed renewal problem in the following elementary form. A population consists of elements which may belong to the population for $1, 2, \cdots, n$ periods; the probability that an element will belong to the population for exactly k periods is supposed to be constant in time and the same for all elements ($\sum_{k=1}^{n} p_k = 1$). In order to keep the size of the population constant, A_r new elements are introduced at the end of the rth period so that $A_r = \sum_{k=1}^{n} A_{r-k} p_k$. The problem consists in determining the asymptotic behavior of the A_k. The author shows that the A_k will approach a limit if and only if there exist two relatively prime integers i and j such that $p_i > 0$ and $p_j > 0$. In that case A_r tends to $A_0\{\sum k p_k\}^{-1}$. The method is similar to that used by the author in the theory of Markov chains [Skand. Aktuarietidskr. **22**, 200–210 (1939); these Rev. **1**, 342]. *W. Feller* (Ithaca, N. Y.).

Nolfi, P. Zur Bestimmung der Rückschlusswahrscheinlichkeit einer geschlossenen Gesamtheit. Mitt. Verein. Schweiz. Versich.-Math. **44**, 217–220 (1944). [MF 13961]

Let $p(\alpha)=\varepsilon^{-\alpha}$ be the probability that a person of a certain age survives a certain period of time; then $\binom{n}{s}(1-\varepsilon^{-\alpha})^s \varepsilon^{-(n-s)\alpha}$ is the probability that, of n such persons, s will die in that period. If actually r deaths occur, what is the "a posteriori probability" $g(\lambda)$ that $0<\alpha<\lambda$? Using Bayes's principle, the author derives

$$g(\lambda) = \sum_{k=r+1}^{n} \binom{n}{k} (1-\varepsilon^{-\lambda})^k \varepsilon^{-(n-k)\lambda},$$

the probability of at least $r+1$ deaths among n persons, computed under the assumption that the probability of survival is $p(\lambda)=\varepsilon^{-\lambda}$. No attempt is made to explain why the use of Bayes's principle is justified.

Z. W. Birnbaum (Seattle, Wash.).

Nolfi, P. Zur mathematischen Darstellung wachsender Gesamtheiten. Mitt. Verein. Schweiz. Versich.-Math. **45**, 311–321 (1945). [MF 15424]

The author considers a stochastic growth process by first introducing an urn model and then passing to the limit. His fundamental assumptions can be more conveniently reformulated by referring from the beginning to a continuous time parameter t. Let the number of individuals in a population (for example, people with a certain contagious disease) be a random variable depending on t. Within any short interval of length dt each individual has a chance $\eta dt + o(dt)$ to give rise to a new individual, while the chance of creating more than one new individual is $o(dt)$; it is assumed that η is a constant and that the existing individuals are mutually statistically independent. Required the probability $p_{ik}(t)$ that within the time interval $(0, t)$ the population will increase from i to k individuals. The author finds, starting from his urn scheme,

$$p_{ik}(t) = \binom{k-1}{i-1} \varepsilon^{-i\eta t} (1-\varepsilon^{-\eta t})^{-(k-i)\eta t}. \tag{1}$$

The same solution could be obtained from the differential equations

$$p'_{ik}(t) = -k\eta p_{ik}(t) + (k-1)\eta p_{i,k-1}(t), \tag{2}$$

which are easily deduced from the above assumptions. [The distribution (1) has been used by Furry in the theory of cosmic rays; cf. N. Arley, On the Theory of Stochastic Processes and their Application to the Theory of Cosmic Radiation, Copenhagen, 1943, pp. 92 ff.; these Rev. **7**, 209. For other applications of (1) and (2) cf. Feller, Acta Biotheoretica Ser. A. **5**, 11–40 (1939); these Rev. **1**, 22; for a derivation of (1) essentially along the author's lines cf. Yule, Trans. Roy. Soc. London, Ser. B. **213**, 21–87 (1924).]

W. Feller (Ithaca, N. Y.).

Kunisawa, Kiyonori. On the power set of a probability law. I. Proc. Phys.-Math. Soc. Japan (3) **24**, 681–695 (1942). [MF 15037]

Two probability distribution functions $F_1(x)$, $F_2(x)$ are said to be in the same class if there are constants a, b with $a>0$ such that $F_1(ax+b)=F_2(x)$. A sequence of classes $K_1, K_2, \cdots$ is said to converge if there are a sequence of functions $F_1(x) \in K_1, F_2(x) \in K_2, \cdots$ and a probability distribution function $F(x)$ not corresponding to a distribution concentrated at a single point, such that $F_n \to F$ at all the continuity points of F. Let $X_1, X_2, \cdots$ be mutually independent chance variables with the common distribution function $F(x)$. The classes determined by the sums $\{S_n = X_1 + \cdots + X_n\}$ constitute the power set of F. The author extends the analysis of the power set made by Doeblin [Studia Math. **9**, 71–96 (1940); these Rev. **3**, 168], using the methods developed by Gnedenko and Groshev [Rec. Math. [Mat. Sbornik] N.S.

6(48), 521–541 (1939); these Rev. 1, 342], finding conditions for classes to be limit classes of the power set of a given F. Using these conditions it is shown that, if both the power set of a distribution function and the derived set of the power set are compact, then the power set is metrizable.
J. L. Doob (Urbana, Ill.).

Esseen, Carl-Gustav. Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. Acta Math. 77, 1–125 (1945). [MF 13767]

This is an exhaustive study of the delicate questions concerning the deviation of the distribution function of a sum of independent random variables from the normal distribution. Apart from new and often striking results a thorough review of the whole subject is also given.

The first two chapters review the most important properties of distribution functions and their characteristic functions. We cite the following two results. (1) If $F(x)$ and $G(x)$ are distribution functions and $f(t)$ and $g(t)$ their respective characteristic functions, then $f(t)=g(t)$ for $|t|\leqq L$ implies $\int_{-\infty}^{\infty}|F(x)-G(x)|dx\leqq\pi/L$. (2) If $f(t)=g(t)$ in an interval about 0 and $\alpha_{2k}=\int_{-\infty}^{\infty}x^{2k}dF(x)<\infty$ then the divergence of $\sum\alpha_{2k}^{-1/2k}$ implies $F(x)=G(x)$ at each point of continuity.

The principal result of chapter 3 consists in an estimate for the difference between the distribution function of $(X_1+\cdots+X_n)/S_n^{\frac{1}{2}}$ (X_i independent random variables each with mean 0 and variance σ_i^2; $S_n=\sigma_1^2+\cdots+\sigma_n^2$) and the normal distribution $\Phi(x)$. The estimate is given in terms of absolute kth moments of X_i, which are assumed to be finite. For $k=3$ the estimate coincides with that found by Berry [Trans. Amer. Math. Soc. 49, 122–136 (1941); these Rev. 2, 228] and independently by the author.

In chapter 4 the author assumes that the X's are identically distributed and proceeds to investigate the asymptotic expansions of the distribution function of $(X_1+\cdots+X_n)/\sigma n^{\frac{1}{2}}$. Cramér's original investigations [Random Variables and Probability Distributions, Cambridge University Press, 1937] were restricted to the case when the characteristic function $f(t)$ of X_i satisfies the condition $\limsup_{t\to\pm\infty}|f(t)|<1$. The author obtains results in two more difficult cases, namely (a) $\limsup_{t\to\pm\infty}|f(t)|=1$ but $|f(t)|<1$ for $t\neq0$ and (b) $f(t_0)=1$ ($t_0\neq0$), in which case $|f(t)|$ is known to be periodic.

In chapter 5 the dependence on both n and x of the remainder in the central limit theorem is investigated. Chapters 6, 7 and 8 are devoted to similar questions in the multidimensional case. There are interesting remarks on the connection between the central limit theorem and the problem of lattice points in an n-dimensional sphere.
M. Kac (Ithaca, N. Y.).

Kunisawa, Kiyonori. Mean concentration function and the law of large numbers. Proc. Imp. Acad. Tokyo 20, 627–630 (1944). [MF 14934]

The author proposes $\psi(l)=l\int_0^{\infty}e^{-lt}|f(t)|^2dt$ as a useful function of mean concentration of a distribution function with characteristic function $f(t)$. It is related to Kawata's function of mean concentration $C(l)$ [Duke Math. J. 8, 666–677 (1941); these Rev. 3, 168] and Lévy's function of maximum concentration $Q(l)$ [Théorie de l'Addition des Variables Aléatoires, Gauthier-Villars, Paris, 1937, § 16] by the inequality $\frac{1}{4}Q(l)^2\leqq C(l)\leqq\psi(l)\leqq(2+\pi)Q(l)$. The value of ψ for the convolution of two distribution functions is at most that for each distribution. Using this function ψ, necessary and sufficient conditions, equivalent to those of Feller [Acta Litt. Sci. Szeged 8, 191–201 (1937)], are found that a series of mutually independent chance variables obeys the generalized law of large numbers. *J. L. Doob.*

Osida, Isao. On the harmonic analysis of random functions. Proc. Phys.-Math. Soc. Japan (3) 24, 292–296 (1942). [MF 15027]

The author discusses the spectrum of a random function, rediscovering the fact that the correlation function is the cosine transform of the spectral density. *J. L. Doob.*

Itô, Kiyosi. On stochastic processes. I. (Infinitely divisible laws of probability). Jap. J. Math. 18, 261–301 (1942). [MF 14969]

The author gives a systematic approach to the study of infinitely divisible probability distributions (that is, distributions which can be expressed as the convolutions of distributions which are arbitrarily near the unit distribution). He defines the mean concentration of a distribution function F by $q_F=\int_0^{\infty}e^{-l}Q(l)$, where $Q(l)$ is Lévy's function of concentration, $\max_x\{F(x+l+0)-F(x-0)\}$. The functional q_F decreases (or remains the same) under the operation of convolution. Distribution functions $\{F(x)\}$ are divided into classes: f is the class containing all distribution functions of the form $F(x+a)$, where $F(x)$ is a fixed distribution function and a is any constant. The classes are partially ordered by the convention that $f\leqq g$ if a distribution function in f is the convolution of one in g with a third distribution function. Finally, two distances are defined in this partially ordered space of classes: $\rho(f,g)$ is defined as the greatest lower bound of the distances between pairs of distribution functions in f and g [in Lévy's sense, Théorie de l'Addition des Variables Aléatoires, Gauthier-Villars, Paris, 1937, p. 47] and $\rho_1(f,g)=|q_F-q_G|$. It is shown that ρ, ρ_1 determine equivalent topologies on any set of classes f which is linearly ordered and which is order bounded from below. Any f set which is order bounded from below is totally ρ bounded. Various known theorems have elegant formulations in terms of these concepts. For example, if $f_1\leqq f_2\leqq\cdots$ or if $f_1\geqq f_2\geqq\cdots\geqq f$, then $\lim_{n\to\infty}f_n$ exists (ρ distance). Using these concepts, the author makes a systematic study of infinitely divisible laws, deducing their (known) relationship to stochastic processes with independent increments and giving a careful derivation of Lévy's formula for their characteristic functions (by way of the jumps of the corresponding processes with independent increments). [See Lévy, op. cit., chapter 7.] *J. L. Doob* (Urbana, Ill.).

Itô, Kiyosi. On the normal stationary process with no hysteresis. Proc. Imp. Acad. Tokyo 20, 199–202 (1944). [MF 14883]

Let the real chance variables $\{x(t)\}$, with t running through all integers or all real numbers, determine a stationary Gaussian process. That is, for any finite set of t values the joint distribution of the corresponding chance variables is Gaussian and is unaffected by translations of the t axis. It is shown that the process is a Markov process if and only if the correlation function, the correlation between $x(s)$ and $x(s+t)$, has the form $e^{-a|t|}$, $a>0$. Normal forms are found for $x(t)$ in this case; neglecting trivial exceptions, if t is integral valued $x(t)$ can always be put in the form $x(t)=\sum_{n=-\infty}^{t}b^{t-n}y(n)$, $|b|<1$, where the chance variables $\{y(n)\}$ are Gaussian and mutually independent; if t is a continuous parameter, $x(t)=\int_{-\infty}^{t}e^{-a(t-\tau)}dy(\tau)$, $a>0$, where $y(\tau)$ determines a Gaussian process with stationary uncorrelated increments (the Einstein Brownian movement process).

Most of the results are known [see Doob, Ann. of Math. (2) **43**, 351–369 (1942); Ann. Math. Statistics **15**, 229–282 (1944); these Rev. **4**, 17; **6**, 89; an expository treatment of this whole subject is given in the latter reference].

J. L. Doob (Urbana, Ill.).

Itô, Kiyosi. On the ergodicity of a certain stationary process. Proc. Imp. Acad. Tokyo **20**, 54–55 (1944). [MF 14870]

Let the real chance variables $\{x(t)\}$ determine a stationary Gaussian process [see the preceding review]. It is shown that the process is ergodic and strongly mixing, that is, that the group of probability preserving translations of the t axis has this property, if and only if the correlation function $\rho(\tau)$ converges to zero when $|\tau|\to\infty$. [The result is stated less generally in the paper, but, as noted by the author in the paper reviewed below, the proof gives the theorem as just formulated.] *J. L. Doob* (Urbana, Ill.).

Itô, Kiyosi. A kinematic theory of turbulence. Proc. Imp. Acad. Tokyo **20**, 120–122 (1944). [MF 14877]

The author points out that in the theory of temporally homogeneous isotropic turbulence a velocity component $u(t)$ at a given point in a given direction at time t determines a stationary stochastic process. If the turbulence is of Gaussian type this stochastic process is a stationary Gaussian process [see the two preceding reviews] and the processes corresponding to orthogonal directions are mutually independent. The theorem quoted in the preceding review then shows that, if the correlation between $u(t)$ and $u(t+\tau)$ converges to zero when $|\tau|\to\infty$, the ergodic theorem can be applied to evaluate the correlation in terms of time means,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u(t)u(t+\tau)d\tau$$

with probability 1. This justifies the standard experimental procedure. *J. L. Doob* (Urbana, Ill.).

Itô, Kiyosi. Stochastic integral. Proc. Imp. Acad. Tokyo **20**, 519–524 (1944). [MF 14918]

Let $g(t,\omega)$ define a Brownian motion ($0\leqq t\leqq 1$, ω varying on a measure space of total measure 1), that is, if $t_1<\cdots<t_n$, the functions (of ω) $g(t_2,\omega)-g(t_1,\omega),\cdots,g(t_n,\omega)-g(t_{n-1},\omega)$ are mutually independent, with Gaussian distributions, means 0, variances $t_2-t_1,\cdots,t_n-t_{n-1}$. Wiener has defined the stochastic integral $\int_0^1 f(\tau)dg(\tau,\omega)$ for $f\in L_2(0,1)$ and shown its fundamental importance [see Paley and Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., vol. 19, New York, 1934, chapter IX]. The author extends this definition to functions $f=f(\tau,\omega)$ satisfying rather weak regularity conditions and such that, for $\sigma>t\geqq\tau$, $f(\tau,\omega)$ is independent of the differences $g(\sigma,\omega)-g(t,\omega)$. *J. L. Doob* (Urbana, Ill.).

Lévy, Paul. Dérivation, intégration et équations différentielles stochastiques. C. R. Acad. Sci. Paris **219**, 602–603 (1944). [MF 15295]

Let $X(t)$ be a random function. The author remarks that, if $X(t)$ has a mean square derivative $\text{l.i.m.}_{h\to 0}\{X(t+h)-X(t)\}/h$ at every point of an interval and $X'(t)$ is continuous in the interval in the mean square sense, $\text{l.i.m.}_{h\to 0}X'(t+h)=X'(t)$, then $X(t)$ can be obtained from $X'(t)$ by integration. This is not necessarily true if $X'(t)$ is obtained as a limit in probability or as a limit with probability 1. The author defines a stochastic differential equation as an equation for the conditional probability law of $X(t+\delta t)-X(t)$ ($\delta t>0$), under the hypothesis $X(t)=x$, exact up to an error which is $o(\delta t)$. If this law depends on the values of $X(u)$ for $u<t$, the equation is called integro-differential. As in ordinary first order differential equations there may be Peano (right) points t_0, that is, points with the property that there is no unique continuation of solutions beyond them. Let $X_1(t)$, $X_2(t)$ be two random functions determined by the given equation, with $X_i(t_0)=x_i$, and suppose that the expectation of $|\delta X_2(t_0)-\delta X_1(t_0)|^2$ does not exceed $(x_2-x_1)^2\varphi(t_0)^2(\delta t)^2$. Then a condition on φ is given which is necessary for t_0 to be a Peano point and sufficient for there to be a stochastic equation with the given φ for which t_0 is a Peano point.

J. L. Doob (Urbana, Ill.).

Fan, Ky. Une définition descriptive de l'intégrale stochastique. C. R. Acad. Sci. Paris **218**, 953–955 (1944). [MF 15242]

The following results are stated without proof. Let $X_k(t)$ be a sequence of random variables defined over the same space and for all values of the time parameter t with $a<t<b$. Suppose that the stochastic integral in the sense of Slutsky of each of the $X_k(t)$ exists and that there is a random variable $X(t)$ such that the expectation of $|X_k(t)-X(t)|$ tends uniformly to zero. Then the stochastic integral of $X(t)$ exists. The variable $X(t)$ is called continuous of order one if to any t_0 and any $\epsilon>0$ there corresponds an $\eta>0$ such that $|t-t_0|<\eta$ implies $E|X(t)-X(t_0)|<\epsilon$. If $X(t)$ is continuous of order one, then it is stochastically integrable. Finally, the stochastic integral is characterized by the properties that $I(X+Y)=I(X)+I(Y)$, that $I(X_k)$ tends in probability to $I(X)$ whenever $E|X(t)-X_k(t)|$ tends uniformly to zero, and that for every $X(t)$ of the form $X(t)=x(t)A$, where $x(t)$ is an ordinary function and A a fixed random variable, the integral reduces to the form A times the Riemann integral of $x(t)$. *W. Feller* (Ithaca, N. Y.).

Nakano, Hidegorô. Über stochastischen Prozess. I. Proc. Imp. Acad. Tokyo **20**, 513–518 (1944). [MF 14917]

The mathematical bases for two stochastic processes depending on a continuous time parameter are outlined. The treatment given, based on the author's work on topological measure [Proc. Phys.-Math. Soc. Japan (3) **25**, 279–334 (1943); these Rev. **7**, 306], is asserted to be simpler than that of the reviewer [Trans. Amer. Math. Soc. **42**, 107–140 (1937)]. The latter treatment was complicated by the difficulty of defining probabilities of events of the type $\text{l.u.b.}_{a\leq t\leq b}x(t)<k$. Enough details are not given by the author to make clear how he has surmounted this difficulty.

J. L. Doob (Urbana, Ill.).

Blanc-Lapierre, André. Sur les fluctuations produites par l'effet de grenaille dans les amplificateurs. C. R. Acad. Sci. Paris **217**, 73–74 (1943). [MF 15230]

The author supposes that a linear amplifier, giving a response $qR(t/\tau)$ to an impulse produced by a charge q acting at the instant $t=0$, is subjected to a succession of such shocks, impressed at a uniform rate ρ, and distributed in time in a Poisson distribution. He then finds the characteristic function of the joint distribution of $x(t_1)$ and $x(t_2)$. When $\rho\tau\to\infty$ this joint distribution becomes Gaussian, with correlation $E\{x(t_1)x(t_2)\}=c\int_{-\infty}^{\infty}R(t)R(t-\theta)dt$, $\theta=t_1-t_2$. The method of calculating higher moments is explained and exemplified by the calculation of $E\{(x(t_1)^2-m_2)(x(t_2)^2-m_2)\}$, $m_2=E\{x(t)^2\}$. The answer is $2q^2I^2\mu_{11}(\theta)^2+q^2I\mu_{22}(\theta)$, where $\mu_{\nu\varepsilon}(\theta)=\int_{-\infty}^{\infty}R(t)^\nu R(t-\theta)^\varepsilon dt$ and $I=\rho q$ is the mean current.

These results make it possible to find the frequency spectrum of $x(t)^2$ and higher powers. *J. L. Doob.*

Blanc-Lapierre, André. Sur certaines fonctions aléatoires introduites par un problème d'électricité. C. R. Acad. Sci. Paris 218, 924–925 (1944). [MF 15240]

[See the preceding review.] Let $x_{t_0}(t)$ be the total effect at time t of all shocks impressed between times t_0 and t. Then $x_{t_0}(t)$ has finite moments of all orders. If $t_0 \to -\infty$ there is a limit with probability 1, $x_{t_0}(t) \to x(t)$, and $x(t)$ determines a stationary process, that is, $x(t)$ probabilities are unchanged by translations of the t-axis. The author supposes that the $R(t)$ of the preceding review and its derivative $R'(t)$ are bounded uniformly for $t \neq 0$, and do not exceed $ce^{-\varphi t}$ for some positive constants c, φ. [The same result was obtained by Khintchine under considerably weaker hypotheses, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 1938, 313–322 (1938).] *J. L. Doob* (Urbana, Ill.).

Blanc-Lapierre, André. Sur quelques propriétés ergodiques de certaines fonctions aléatoires. C. R. Acad. Sci. Paris 218, 985–986 (1944). [MF 15247]

[See the preceding review.] Let $f(x)$ be a function of the real variable x. The author proves, for various functions f, that $\lim_{T\to\infty} T^{-1}\int_0^T f\{x(t)\}dt = E\{f[x(t)]\}$ and that for $f = x^p$, p a positive integer, this integral average is asymptotically Gaussian for large T. [A result essentially the same as the first result was obtained by Khintchine [reference in the preceding review] as an application of Birkhoff's ergodic theorem; Khintchine also noted the second result in the case $p = 1$.] *J. L. Doob* (Urbana, Ill.).

Blanc-Lapierre, André. Les fonctions aléatoires stationnaires laplaciennes. C. R. Acad. Sci. Paris 220, 378–380 (1945). [MF 13949]

The author has shown in the three papers reviewed above that the shot effect in vacuum tubes leads to the study of stationary Gaussian stochastic processes to describe the current and voltage fluctuations in vacuum tube circuits. In the present paper he discusses the fundamental role of the serial correlation function, and makes the following general remark: in many applications a system described by p functions $g_1(t), \cdots, g_p(t)$, whose development is determined by a dissipative system of linear differential equations with constant coefficients, is subjected to random Gaussian perturbations of such a character that $g_1(t), \cdots, g_p(t)$ then determine a stationary p-dimensional Gaussian process. [For further details in this connection see Doob, Ann. Math. Statistics 15, 229–282 (1944); Wang and Uhlenbeck, Rev. Modern Phys. 17, 323–342 (1945); these Rev. 6, 89; 7, 130.] *J. L. Doob* (Urbana, Ill.).

Sarymsakov, T. A. Sur les chaines de Markoff à une infinité dénombrable d'états possibles. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 617–619 (1945). [MF 14414]

The algebraic results of the author's paper in the same C. R. (N.S.) 47, 326–328 (1945) [these Rev. 7, 233] are used to derive in a simple way known ergodic properties of infinite discrete Markov chains. *W. Feller* (Ithaca, N. Y.).

Bogolyubov, N. N. On certain limiting distributions for sums depending on arbitrary phases. Uchenye Zapiski Moskov. Gos. Univ. Fizika 77, 43–50 (1945). (Russian) [MF 15108]

A particular case of the multidimensional central limit theorem is proved in detail. No references to the literature are given. *M. Kac* (Ithaca, N. Y.).

Bogolyubov, N. N. On the influence of a random force on a harmonic oscillator. Uchenye Zapiski Moskov. Gos. Univ. Fizika 77, 51–73 (1945). (Russian) [MF 15109]

The author considers the equation $q'' + k^2 q = f(t)$, where $f(t) = \sum a_j \cos(\omega_j t + \varphi_j)$ and $\varphi_1, \varphi_2, \cdots$ are independent random variables uniformly distributed in $(0, 2\pi)$. Various limiting distributions related to $q(t)$ and $q'(t)$ are derived (using the result of the paper reviewed above) and discussed. Most of the results are well known in the theory of Brownian motion and random noise [see, for instance, M. C. Wang and G. E. Uhlenbeck, Rev. Modern Phys. 17, 323–342 (1945); these Rev. 7, 130; where numerous references are given]. The present paper seems to have been intended as an exposition. *M. Kac* (Ithaca, N. Y.).

Lévy, Paul. Une propriété d'invariance projective dans le mouvement brownien. C. R. Acad. Sci. Paris 219, 378–379 (1944). [MF 15266]

Let $x(t)$ be the random function of a Brownian movement, defining a Gaussian process with $E\{x(t)\} = 0$, $E\{[x(t) - x(s)]^2\} = |t - s|$. The author discusses the Gaussian variable $\xi(t)$, of unit variance, defined by $x(t) = \mu(t) + \sigma(t)\xi(t)$, where $a < t < b$ and $\mu(t)$ is the conditional expectation of $x(t)$ for given $x(a)$ and $x(b)$. The correlation coefficient $\rho(t_0, t_1)$ of $\xi(t_0)$ and $\xi(t_1)$, $t_0 < t_1$, is given by

$$\rho^2 = (t_0 - a)(b - t_1)/(t_1 - a)(b - t_0).$$

The stochastic properties of $\xi(t)$ in the interval (a, b) are therefore invariant under projective transformations of the line, if only the point at ∞ does not go into the interior of the transformed interval. In particular, if b is made infinite it turns out that theorems on the conduct of $x(t)$ for known values of $x(a)$ and $x(b)$ reduce to theorems on $x(t)$ for known values of $x(a)$. This fact has been applied by the author in another paper [Comment. Math. Helv. 16, 242–248 (1944); these Rev. 6, 5]. *J. L. Doob* (Urbana, Ill.).

Vladimirsky, V., and Terletsky, J. Hydrodynamical theory of translational Brownian motion. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 15, 258–263 (1945). (Russian. English summary) [MF 15381]

The authors replace the Langevin differential equation governing the motion of a Brownian particle, $m\ddot{\xi}(t) + b\dot{\xi}(t) = f(t)$, where m is the mass of the particle, $b\dot{\xi}$ is the frictional resistance, and $f(t)$ a random function, by a differential equation, in which the Stokes resistance term $b\dot{\xi}$ is replaced by an expression involving the whole past of $\xi(t)$, due to Boussinesq [Cours de Physique Mathématique, vol. 2, Paris, 1903, p. 238]:

$$m'\ddot{\xi}(t) + b(m'b/\pi)^{\frac{1}{2}} \int_0^\infty \ddot{\xi}(t - \beta^2)d\beta + b\dot{\xi}(t),$$

where m' is half the mass of the fluid displaced by the particle. According to a theorem of Vladimirsky [same Zhurnal 12, 199–202 (1942); these Rev. 4, 208], the dispersion of $\xi(t) - \xi(0)$ is equal to the path traversed by a particle subjected to the same conditions except that the random force [whose properties are not described in detail in the present paper] is replaced by $2kT$; the particle is supposed to be initially at rest. Using this theorem, the dispersion of $\xi(t) - \xi(0)$ is evaluated and shown to approximate the known expressions of Einstein and of Ornstein under suitable restrictions on t and on the parameters concerned.

J. L. Doob (Urbana, Ill.).

Kakutani, Shizuo. On Brownian motions in n-space. Proc. Imp. Acad. Tokyo **20**, 648–652 (1944). [MF 14937]

The one-dimensional Brownian motion process discussed by the author is the usual process, with Gaussian stationary independent increments, determined by chance variables $x(t)$, $-\infty<t<\infty$, with the property that, for every integer r, if $t_1<\cdots<t_r$, the chance variables $x(t_2)-x(t_1)$, $\cdots$, $x(t_r)-x(t_{r-1})$ are mutually independent and Gaussian, with zero means and dispersions t_2-t_1, $\cdots$, t_r-t_{r-1}. The motion in n-space is determined by the parametric equations $x_j=x_j(t)$, $j=1, \cdots, n$, where each $x_j(t)$ process is as just defined, and where the processes are mutually independent. The following theorems are proved. (1) For $n\geqq 5$, almost all paths have no double points. [Essentially this result was proved by Ville for $n\geqq 3$, C. R. Acad. Sci. Paris **215**, 51–52 (1942); these Rev. **5**, 125.] (2) For $n\geqq 3$, almost all paths tend to ∞ as $t\to\infty$. (3) For $n\geqq 3$, almost all paths are nowhere dense. *J. L. Doob* (Urbana, Ill.).

Kakutani, Shizuo. Two-dimensional Brownian motion and harmonic functions. Proc. Imp. Acad. Tokyo **20**, 706–714 (1944). [MF 14946]

The author considers two-dimensional Brownian motions [cf. the preceding review] and proves the following theorems. (1) Let D be a domain bounded by a finite number of disjoint Jordan curves. Let E be a subset of the boundary of D, consisting of a finite number of disjoint Jordan arcs. Then the probability that the Brownian motion starting from Q in D will enter E for some $t>0$ without previously meeting boundary points not in E is a bounded harmonic function of Q in D, taking on the boundary value 1 at inner points of E and 0 at inner points of the complementary boundary arcs. (2) Let F be a compact plane set whose complement is connected. Let D be a finite domain containing F and bounded by a Jordan curve J. Let $\varphi(Q)$ be the probability that the Brownian motion starting at Q in D but not in F will enter F for some $t>0$ without previously meeting J. Then $\varphi(Q)$ is the harmonic measure [see Nevanlinna, Eindeutige Analytische Funktionen, Springer, Berlin, 1936] of F with respect to the domain $D-F$ and the point Q. If F is of positive capacity, $\varphi(Q)$ is harmonic and positive in $D-F$. Moreover, $\varphi(Q)$ takes on the boundary value 0 on J, and, if every neighborhood of some point Q_0 in F is of positive capacity, $\limsup_{Q\to Q_0}\varphi(Q)=1$. (3) Let F be a closed plane set with an inner point. Then for any Q not in F the probability that the Brownian motion starting from Q will enter F for some $t>0$ is 1. More generally, if F is any compact plane set of positive capacity, the probability is 1 that the Brownian motion starting from Q will enter F infinitely many times for infinitely large t. (4) Almost all paths are everywhere dense in the plane and return to each neighborhood of any given point infinitely many times for infinitely large t. This shows that the last result quoted in the preceding review is not true for $n=2$.

J. L. Doob (Urbana, Ill.).

Milatz, J. M. W. Brownian motion. Nederl. Tijdschr. Natuurkunde **8**, 19–36 (1941). (Dutch)

Expository lecture.

Osida, Isao. Brownian motion in a plasto-elastic medium. Proc. Phys.-Math. Soc. Japan (3) **24**, 599–601 (1942). [MF 15035]

The author proposes that the usually accepted correlation function $E\{u(s)u(s+t)\}=\epsilon^{-\beta|t|}$ between Brownian movement velocities at times s and $s+t$ in a viscous medium should be modified to $E\{u(s)u(s+t)\}=\epsilon^{-\beta|t|}\cos\omega t$ if the medium is plasto-elastic. Using this correlation function, he evaluates the dispersion of the displacement:

$$E\{[x(t)-x(0)]^2\}=\frac{2E\{[u(t)]^2\}}{\beta^2+\omega^2}\{\beta t-[\cos\epsilon-\epsilon^{-\beta t}\cos(\omega t+\epsilon)]\},$$

where $t>0$, $\tan\epsilon=-2\beta\omega/(\beta^2-\omega^2)$. *J. L. Doob.*

Osida, Isao. Statistical theory of transport phenomena. Proc. Phys.-Math. Soc. Japan (3) **25**, 590–594 (1943). [MF 15071]

Starting from the distribution function for the random flight of a single gas-molecule, the author finds again the results of the crude theory of transport phenomena which makes use of the mean free path. *F. Zernike.*

Mou, Tchen Chan. Some mathematical considerations which play a role in diffusion problems. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 400–410 (1944). (Dutch. German, English and French summaries) [MF 15363]

The author gives a general discussion of the Fokker-Planck partial differential equations for the transition probability functions of Markov processes and the related equations for the reverse probabilities (motion backwards in time), paying particular attention to processes for which there is a constant density for the probability that the state at time t is y. In this case he finds that if the expectation of increment dy in time dt is $\psi(y,t)dt$ and the expectation of the square of this increment is $2\varphi(y,t)dt$, then $\psi=\partial\varphi/\partial y$. This leads to a slight simplification of the Fokker-Planck equations. *J. L. Doob* (Urbana, Ill.).

Dedebant, G., et Wehrlé, Ph. Mécanique aléatoire. I. Le calcul aléatoire (suite). II. Applications physiques. Portugaliae Phys. **1**, 179–296 (1945). [MF 15219]

After a brief conclusion to the mathematical part [Portugaliae Phys. **1**, 95–149 (1944); these Rev. **7**, 129], consisting of a table of errata and three annexes, the authors devote part II to the application of their theory to physical phenomena. They are chiefly concerned with mechanics of particles, systems and deformable media, together with the thermodynamical concepts and principles needed to make the equations of deformable media complete. Stating that classical mechanics reposes on the concept of the mass particle, the authors propose to substitute for it the "statistical particle," namely, a set of three stochastic functions of the time parameter t; thus the building-stone of their theory is the triple $(X/t, Y/t, Z/t)$, augmented by derivatives of various orders of these stochastic functions, as defined in part I. The authors parallel the variational principles of mechanics by variational principles involving time integrals of means of functions of those stochastic functions which are the variables of state of the system. By an elaborate use of the formalism of part I, together with additional assumptions often made ad hoc, a series of theorems are outlined which parallel to a considerable extent those of the classical theory. The whole approach is to be regarded as an exploration of the possibilities of the new formalism, rather than a definitive proof that it offers any real advantage over the classical treatment. *B. O. Koopman.*

Kolmogorov, A. N., Editor. Collection of papers on the theory of artillery fire. I. Trav. Inst. Math. Stekloff **12**, 106 pp. (1945). (Russian) [MF 14577]

The authors set up the basic statistical assumptions underlying artillery fire, and direct their attention to the

cases in which (intentional) dispersion of fire is preferred to repeated fire at the same point. They find the optimum type of dispersion of fire under various assumptions on the dimensionality of the target (a function of the type of target and type of projectile) and on the relative sizes of the various expected errors. *J. L. Doob* (Urbana, Ill.).

Massé, Pierre. Sur les principes de la régulation d'un débit aléatoire par un réservoir. C. R. Acad. Sci. Paris **219**, 19–21 (1944). [MF 15249]

Massé, Pierre. Sur les effets de la régulation d'un débit aléatoire par un réservoir. C. R. Acad. Sci. Paris **219**, 150–151 (1944). [MF 15250]

Massé, Pierre. Sur un cas particulier remarquable de la régulation d'un débit aléatoire par un réservoir. C. R. Acad. Sci. Paris **219**, 173–175 (1944). [MF 15251]

Suppose that the influx of water into a reservoir is a random variable with known probability distribution. On the other hand, water can be drained from the reservoir in an arbitrary manner and the utility of any possible flow and of the potential reserves in the reservoir are supposed to be known functions. With any fixed plan of utilization, the total utility thus becomes a random variable. The author shows in great generality how a plan can be devised so that the expected value of the utility will become a maximum. The method is applied to a special example, but the actual conditions imposed are too complicated to be described here. *W. Feller* (Ithaca, N. Y.).

Deutsch, Armin J. The probability distribution around a fix in celestial navigation. Amer. J. Phys. **13**, 379–383 (1945). [MF 15373]

Mathematical Statistics

★**Linder, Arthur. Statistische Methoden für Naturwissenschafter, Mediziner und Ingenieure.** Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, 6. Mathematische Reihe, Band III. Verlag Birkhäuser, Basel, 1945. 150 pp.

This monograph is written to serve a twofold purpose. First, it is to acquaint the applied statistician with the use of sampling distributions and tests of significance. Moreover, it intends to give a brief introduction to the mathematical theory behind these statistical methods. This double purpose imposes serious limitations upon the selection of the topics discussed. The first chapter gives definitions of some important statistics, such as the mean, the variance, the regression coefficient and the simple correlation coefficient. In the second chapter some tests of significance for these statistics as well as the chi-square test of goodness of fit are discussed. The first two chapters are written for the applied statistician, no mathematical derivations being given; the formulae used are stated and their application is explained by numerical illustrations. The third chapter is devoted to a brief exposition of the underlying theory. The attempt to take care of the needs of the applied and the theoretical statistician in a book of 150 pages leaves too little space for the mathematical part. Short as it is, it might stimulate the curiosity of the mathematical reader but not satisfy it. Due to the limitations mentioned, only the bivariate normal distribution is considered and, accordingly, partial and multiple correlation coefficients are not discussed. The distribution of the simple correlation coefficient is given only for large samples. Regression is defined as the linear function fitted by least squares; the regression function as an expectation in the conditional distribution is not mentioned. Analysis of variance, as a statistical technique, is not discussed; however, it is applied to testing the linearity of regression, without indicating the wider possibilities inherent in this method. The theory of testing hypotheses and the theory of estimation are only somewhat vaguely mentioned in the introduction. While these omissions may be explained by the aims of the book, there are a few other details objectionable to the mathematically minded reader. The distinction between a statistic and a population parameter could be stated more clearly. The proof of the reproductive property of the chi-square distribution is not carried out rigorously. The series for the probability integral of the chi-square distribution is only given for even degrees of freedom. The discussion of the sign of the correlation coefficient is rather awkward. Some space is devoted to demonstrating that the normal distribution may be obtained as the limit of a binomial distribution with $p=q=0.5$. This appears to be unnecessary and even incongruous with the aims of the book, especially as it is not mentioned that this is only a particular case of a general theorem. The usefulness of the book could have been increased by cross references between chapters II and III and by a more complete list of references. *E. Lukacs* (Cincinnati, Ohio).

Vajda, S. On the constituent items of the reduction and the remainder in the method of least squares. Ann. Math. Statistics **16**, 381–386 (1945). [MF 15471]

Given a set of variates y_i $(i=1, \cdots, n)$ normally and independently distributed, the set having variance 1, and given also a matrix (x_{ik}) $(i=1, \cdots, n; k=1, \cdots, s)$, of rank s, let $\psi^2_{\min}$ denote the minimum (in terms of $b_1, \cdots, b_s$) of $\psi^2=\sum_i(y_i-\sum_k x_{ik}b_k)^2$. R. A. Fisher proved [Metron **5**, no. 3, 90–104 (1925)] that it is possible to express $\psi_{\min}$ as the sum of $n-s$ squares of linear functions of the y_i. This paper gives an explicit formula for n linear functions of the y_i, mutually orthogonal and normalized. The sum of the first s squares (corresponding to the degrees of freedom) is equal to $\sum y_i^2-\psi^2_{\min}$, leaving the remaining $n-s$ squares to yield a sum equal to $\psi^2_{\min}$ as desired. Some interesting special cases are worked out in detail. *A. A. Bennett.*

Masuyama, Motosaburô. Correlation coefficient between two sets of complex vectors. Proc. Phys.-Math. Soc. Japan (3) **23**, 918–924 (1941). [MF 15012]

The usual notions leading to the notion of a correlation coefficient are generalized (i) to complex numbers and (ii) to complex vectors. The usual sums $(x_1(t), x_2(t)$ real$)$ $\sum(x_1(t))^2$, $\sum x_1(t)x_2(t)$ are replaced by $\sum z_1(t)z_1^*(t)$, $\sum z_1(t)z_2^*(t)$ when $z_1(t)$ and $z_2(t)$ are complex, where z^* is the conjugate of z, and by $\sum Z_1(t)Z_1^*(t)$, $\sum Z_1(t)Z_2^*(t)$, when $Z_1(t)$ and $Z_2(t)$ are vectors with complex coefficients for each fixed t, and the products are dyadics (in the sense of Gibbs). Invariance properties are discussed, and the treatment is related to earlier work of the author [same Proc. (3) **23**, 194–195, 196–199, 199–204, 346–351, 351–355 (1941); these Rev. **3**, 6] and Sverdrup [no reference given]. *J. W. Tukey* (Princeton, N. J.).

Masuyama, Motosaburô. On the significance test of the additive correlation coefficient. Proc. Phys.-Math. Soc. Japan (3) **23**, 1016–1019 (1941). [MF 15016]

Testing the significance of the author's additive correlation coefficient [cf. the preceding review] is shown to be

reducible to a standard analysis of variance test (Fisher's z or Snedecor's F). *J. W. Tukey* (Princeton, N. J.).

Wold, Herman. A theorem on regression coefficients obtained from successively extended sets of variables. Skand. Aktuarietidskr. **28**, 181–200 (1945). [MF 14433]

A theorem concerning multiple regressions is proved. Using the customary notation and terminology, let ξ_0 be the dependent variable and let $\{\xi_i\}$ $(i=1, 2, \cdots)$ be a sequence of independent variables. Let r_{ij} denote the correlation between ξ_i and ξ_j and let $b_{01.23\cdots n}$ be the partial regression coefficient of ξ_0 on ξ_1 when the first n explaining variables $\xi_1, \cdots, \xi_n$ are included in the analysis. First it is shown that the point $(r_{0,n+1}, r_{1,n+1}, \cdots, r_{n,n+1})$ must lie inside a certain ellipsoid E determined by the correlation matrix of $\xi_0, \cdots, \xi_n$. The main result is that the volume of the part of E for which $b_{01.23\cdots n} < b_{01.23\cdots n+1}$ is equal to the volume of the part of E for which $b_{01.23\cdots n} > b_{01.23\cdots n+1}$. *A. Wald.*

Baer, Reinhold. Sampling from a changing population. Ann. Math. Statistics **16**, 348–361 (1945). [MF 15468]

Let $x(t)$ be a chance variable which depends on a continuous parameter t, $0 \leqq t \leqq 1$ (or t may lie in any other finite closed interval). Let the expected value of $x(t)$ be denoted by $a(t)$, and let $M_i(t)$ denote the ith order moment of $x(t)$ about $a(t)$. It is assumed that, if $t_1, \cdots, t_n$ are any system of different t's, the corresponding x's are independently distributed. Most of the results of the paper are concerned with the stochastic convergence of certain statistics which can be used as moment estimates, and the remainder deal with the limits of the variances of several of them. Typical results are as follows. Suppose that $a(t)$, $M_2(t)$, $M_3(t)$, $M_4(t)$ exist and are continuous. For every positive integral n let there be a given sequence $t_1, \cdots, t_n$ such that t_j lies between $(j-1)/n$ and j/n $(j=1, \cdots, n)$. Then

$$d^2 = (2n)^{-1} \sum_{j=1}^{n-1} \{x(t_j) - x(t_{j+1})\}^2$$

converges stochastically to $\int_0^1 M_2(t)dt$, and

$$s^2 = (n^{-1}) \sum_{j=1}^{n} \{x(t_j) - \bar{x}\}^2,$$

where $\bar{x} = n^{-1} \sum_{j=1}^{n} x(t_j)$, converges stochastically to

$$\int_0^1 \{M_2(t) + (a(t) - a)^2\} dt,$$

where $a = \int_0^1 a(t)dt$. *J. Wolfowitz* (Raleigh, N. C.).

Kawata, Tatsuo. The distribution of grouped moments in large samples. Proc. Imp. Acad. Tokyo **20**, 337–339 (1944). [MF 14897]

By direct computation and use of Laplace's theorem the author establishes the following theorem. Suppose that the population variable has a finite $2r$th moment and let the sample moment of the rth order formed from a sample of size n of the population be ${}_1M_r$. Then $n^{\frac{1}{2}}({}_1M_r - E({}_1M_r))$ converges in distribution to the normal law with mean 0 and variance $E({}_1M_{2r}) - (E({}_1M_r))^2$. Here E is the usual expectation arising from the given probability density.

A. A. Bennett (Providence, R. I.).

Cheriyan, K. C. Distributions of certain frequency constants in samples from non-normal populations. Sankhyā **7**, 159–166 (1945). [MF 15135]

Samples from χ^2-distributions with 2, 4 and 9 degrees of freedom were used to construct three bivariate populations with $\rho = 1/2$, $3/4$ and $8/9$, respectively. Two hundred samples of 5 were taken from each of these populations to test the agreement of the distribution of r and z with "normal theory." The agreement for r was not bad except for a large excess of observed over expected for values of $r > .9$. The agreement was somewhat better for z. Observed and expected values of β_1 and β_2 were given, but no attempt was made to test these for significant departures from expectation. The first two moments and betas were computed for the means and variances of these 200 samples, using, for the latter, formulas given by Tchouproff and Church. The mean variance was smaller than expected in populations I and III, but no test of significance was given. The agreement for the means was in general quite good.

R. L. Anderson (Raleigh, N. C.).

Roy, Samarendra Nath. The individual sampling distribution of the maximum, the minimum and any intermediate of the p-statistics on the null-hypothesis. Sankhyā **7**, 133–158 (1945). [MF 15134]

Let S' be a sample of size n' drawn from a p-variate normal population Σ' and S'' a sample of size n'' drawn from a p-variate normal population Σ''. Denote by $\|a'_{ij}\|$ the variance-covariance matrix of S' and by $\|a''_{ij}\|$ that of S''. The roots $k_1^2, \cdots, k_p^2$ of the determinantal equation $|a'_{ij} - k^2 a''_{ij}| = 0$ play an important role in problems of multivariate statistical analysis. The joint distribution of these roots was obtained some time ago by the author and by others under the assumption that the variance-covariance matrix of Σ' is equal to that of Σ''. In this paper the author derives the distribution of the maximum, the minimum and any intermediate root. These distributions can be expressed as p-fold integrals. These integrals are evaluated with the help of a recursion formula reducing them step by step to one-fold integrals. *A. Wald* (New York, N. Y.).

Derksen, J. B. D. Probability-theoretical foundations of "regression-analysis." Nederl. Tijdschr. Natuurkunde **8**, 37–54 (1941). (Dutch)

Expository lecture.

Reiersöl, Olav. Confluence analysis by means of instrumental sets of variables. Ark. Mat. Astr. Fys. **32A**, no. 4, 119 pp. (1945). [MF 15176]

This monograph is an outgrowth of a line of attack on confluence analysis problems initiated by the author in his paper in Econometrica **9**, 1–24 (1941) [these Rev. **2**, 237]. It is assumed that among a set of quantitatively measurable variables (usually economic in character) there exist linear structural relations. By means of observations upon these variables it is undertaken to limit within as small an angle as possible in a Euclidean space the vector whose components are the coefficients in the structural equation. The essential feature of the methods developed here is the introduction of another set of variables, the instrumental set, equal in number to the set being investigated and bearing a usefully close relation to them. For example, the instrumental set may be the investigational set with a time lag, they may be moving averages of the investigational set or residuals after a trend has been removed, they may be direct functions of corresponding investigational variables or, to include previous treatments, they may be identical with the investigational set. The methods of analysis are developed from a study of the cross moment matrix of all covariances between a member of one set and a member of the other. Each investigational variable is taken as the sum

of a structural part and a disturbance and for the most part it is assumed that a disturbance in an investigational variable is uncorrelated with a noncorresponding instrumental variable. The actual matrix studied is the one for cross moments between structural variables and instrumental variables. An extended body of results is obtained, including the examination of a variety of special cases, the description of which would be too lengthy to give here. There is also extended discussion of the applicability of the methods developed and there are two illustrative numerical examples. *C. C. Craig* (Ann Arbor, Mich.).

Reiersöl, Olav. Residual variables in regression and confluence analysis. Skand. Aktuarietidskr. **28**, 201–217 (1945). [MF 14434]

In a previous study of confluence analysis [see the preceding review] the author introduced the use of an instrumental set of variables Y equal in number to that of the set X under investigation. The principal result here is that instrumental regressions in the set X with Y's as the instrumental set are the same whether the X's or the residuals in the instrumental regressions are used. If the instrumental set is identical with the investigational set, the result affords a generalization of the main theorem of a paper by Frisch and Waugh [Econometrica **1**, 387–401 (1933)]. It is then shown how this result can be used to facilitate computations in confluence analysis. *C. C. Craig* (Ann Arbor, Mich.).

van Uven, M. J. Likelihood as conditioned probability. Nederl. Akad. Wetensch., Proc. **44**, 947–955 (1941). [MF 15762]

If an a priori distribution of unknown parameters is known, these parameters can be estimated from a sample by choosing those parameters which are most probable in the light of the given sample. This solution of the estimation problem leads to the often discussed difficulties involved in the fact that either there may be no a priori distribution of the parameters or, if there is one, it is unknown, and unfortunately this solution of the estimation problem depends essentially on the a priori distribution assumed. The author removes the latter ambiguity by choosing a particular set of variables to determine the a priori distribution of the parameters, variables whose own density of distribution is identically constant. With these variables, the method is formally completely identical with R. A. Fisher's method of maximum likelihood. Some of the formulas of that method are obtained, with no attempt at mathematical rigor.
J. L. Doob (Urbana, Ill.).

Itô, Kiyosi. On Student's test. Proc. Imp. Acad. Tokyo **20**, 694–700 (1944). [MF 14944]

Proofs of the well-known theorems that the Student tests for a mean and for the difference of two means are uniformly most powerful against one-sided alternatives under the usual assumptions of normality, equal variance of all observations, etc. Although the basic concepts of the Neyman-Pearson theory are used, there is no mention of these writers or their proofs of these theorems [Philos. Trans. Roy. Soc. London. Ser. A. **231**, 289–337 (1933)]. *H. Scheffé.*

Guttman, Louis. A basis for analyzing test-retest reliability. Psychometrika **10**, 255–282 (1945). [MF 14659]

The author considers the scores t_{ik} of individuals i on tests k, and observes that [reviewer's notation]

$$\underset{ik}{E}(t_{ik}-t_{..})^2=\underset{i}{E}(t_{i.}-t_{..})^2+\underset{k}{E}\,\underset{i}{E}(t_{ik}-t_{i.})^2,$$

where E represents the expected (average) value and dots also indicate averages over the replaced subscript. In general, the ratio of the first term on the right to the whole serves as a precise definition of the reliability coefficient of the test. If the test is divided into items, and if it is scored by adding item scores, $t_{ik}=\sum x_{ijk}$, where j designates items. The author establishes six lower bounds for the reliability coefficient of the test under the hypotheses that (A) the item scores have four finite moments, (B) i and k each run over indefinitely large ranges, (C) the first four mixed moments of $(x_{hgk}-x_{hg\cdot})$, $(x_{igk}-x_{ig\cdot})$, $(x_{Hjk}-x_{Hj\cdot})$, $(x_{Ijk}-x_{Ij\cdot})$, where the pairs (h, g), (i, g), (H, j), (I, j) are distinct, can be calculated from the moments of the separate variables as in the independent case.

Using the same hypotheses, it is shown that the results of a single application of the test to an infinite population would allow the calculation of these six lower bounds. One of the simplest of these bounds is $L_4=2\{1-(S_a^2+S_b^2)/S_t^2\}$, where S_a^2 and S_b^2 are the variances of the scores on two "halves" of the test and S_t^2 is the variance of the total score.
J. W. Tukey (Princeton, N. J.).

Fisher, R. A. The logical inversion of the notion of the random variable. Sankhyā **7**, 129–132 (1945). [MF 15133]

Criticizing the Neyman theory of tests of significance, the author objects to "laying down axiomatically, what is not agreed or generally true, that the level of significance must be equal to the frequency with which the hypothesis is rejected in repeated sampling of any fixed population allowed by the hypothesis." "The purpose of this note is therefore to discuss . . . the process of reasoning by which we may pass . . . from forms of statement in which observations are regarded as random variables . . . to forms of statement in which the observations constitute fixed data, and frequency distributions are found for the unknown parameters regarded as random variables." *W. Feller.*

Tintner, Gerhard. Multiple regression for systems of equations. Econometrica **14**, 5–36 (1946). [MF 15534]

Let X_{it} $(i=1, \cdots, p; t=1, \cdots, N)$ be a system of p statistical variables as observed at N time points. The author is concerned with the causal structure of the system. In line with C. Gini [Metron **1**, no. 3, 63–82 (1921)], R. Frisch [Oslo Univ. Økon. Inst. Publ. no. **5**, 1934] and others, the author assumes that the structure is of the following pattern: (α) each X_{it} allows the representation $X_{it}=M_{it}+y_{it}$, where the "disturbances" y_{it} are normally distributed and independent of each other and of the "systematic parts" M_{it}, and where the variance of y_{it} does not vary with t; (β) the M_{it} are connected by R exact linear relations $k_{v0}+\sum_{j=1}^{p}k_{vj}M_{jt}=0$ $(v=1, \cdots, R; t=1, \cdots, N)$. The author's problem is to estimate the constants k_{vj}. The new twist of the treatment consists in first estimating the variances of the y_{it} (by the variate difference method or otherwise), next estimating R by the method of P. L. Hsu [Ann. Eugenics **11**, 39–41 (1941); these Rev. **3**, 8], the estimates thus obtained being generally compatible with a multiple infinity of systems of type (β). The system is made unique by assuming a priori that certain k_{vj} are zero, the method of maximum likelihood then leading to linear equation systems for estimating the remaining coefficients k_{vj}. The procedure is illustrated by determining a demand curve and a supply curve from a set of five time series. *H. Wold.*

Gumbel, E. J. La durée de retour. La Météorologie **1942**, 71–98 (1943). [MF 15551]

This is one of a series of papers in which the author has studied the behavior of extreme values and, in particular, the laws governing their recurrence. The reader should refer to at least Ann. Inst. H. Poincaré **4**, 115–158 (1935); Ann. Math. Statistics **12**, 163–190 (1941); these Rev. **3**, 10. Here the characteristics of the distribution of values in excess of a given value are discussed, together with the methods of estimation from observed data. There is an examination of the forms assumed by this distribution for several underlying probability density functions of individual values. The second section is devoted to a study of the distribution of the extreme value in a sample of given size and is largely a review of results and methods developed by the author in earlier papers. These characterize such a distribution and give means of applying it to meteorological and hydrological data, in particular. *C. C. Craig* (Ann Arbor, Mich.).

Harshbarger, Boyd. On the analysis of a certain six-by-six four-group lattice design using the recovery of interblock information. Ann. Math. Statistics **16**, 387–390 (1945). [MF 15472]

In his previous paper [Ann. Math. Statistics **15**, 307–320 (1944); these Rev. **6**, 93] the author did not utilize the interblock information. In this paper he gives formulae which utilize the interblock information. *H. B. Mann.*

Mathematical Biology

Opatowski, I. Chain processes and their biophysical applications. Bull. Math. Biophys. **7**, 161–180 (1945). [MF 14535]

Under exposure to radiant energy biological units undergo changes. The present paper develops the mathematical theory of such changes, under the assumption that there are $n+1$ possible states, and that passages are possible only from state i to $i+1$ (direct process, injury) or from state i to state $i-1$ (reverse process, recovery). The equations are then $dY_i/dt=k_iY_{i-1}-k_{i+1}Y_i+g_iY_{i+1}-g_{i-1}Y_i$, where the k_i, g_i are constants (which will depend on experimental conditions). In many biological situations only the final state can be observationally distinguished from the remainder. It is shown that under these circumstances a single experiment cannot establish whether or not recovery occurs.

C. P. Winsor (Baltimore, Md.).

Opatowski, I. Chain processes and their biophysical applications. II. The effect of recovery. Bull. Math. Biophys. **8**, 7–15 (1946). [MF 15593]

Continuing the paper reviewed above, an expression is given for the probability that an organism is in the state n at time t if it was in the state 0 at $t=0$. *C. P. Winsor.*

Malécot, Gustave. Sur un problème de probabilités en chaine que pose la génétique. C. R. Acad. Sci. Paris **219**, 379–381 (1944). [MF 15267]

In a constant population of N individuals let a Mendelian gene at a certain moment occupy r loci; required the probability that after n generations it will occupy s loci (panmixia and absence of selection and mutation being assumed). It is shown that the required probabilities form a simple Markov chain and that by studying the characteristic roots more precise information as to the asymptotic behavior can be obtained than is furnished by the known solutions of the problem. *W. Feller* (Ithaca, N. Y.).

Malécot, Gustave. La diffusion des gènes dans une population mendélienne. C. R. Acad. Sci. Paris **221**, 340–342 (1945). [MF 14504]

If homogamy, selection, and recurrent mutations are taken into account, the determination of the gene distribution in successive generations reduces to a problem in Markov chains. The author in this way obtains a limiting distribution which is said to have been mentioned by R. A. Fisher and by S. Wright. *W. Feller* (Ithaca, N. Y.).

Wright, Sewall. The differential equation of the distribution of gene frequencies. Proc. Nat. Acad. Sci. U. S. A. **31**, 382–389 (1945). [MF 14530]

The author derives differential equations satisfied by the probability density function of the distribution of gene frequencies under the following conditions: (1) stationary distribution of gene frequencies, (2) steady flow due to mutation pressure from gene frequency 0 to gene frequency 1 or vice versa, (3) nonstationary states. Under conditions (1) and (2) and in special cases under condition (3) the author solves the respective differential equations explicitly.

H. B. Mann (Columbus, Ohio).

Geiringer, Hilda. Further remarks on linkage theory in Mendelian heredity. Ann. Math. Statistics **16**, 390–393 (1945). [MF 15473]

Supplementing a previous paper [same Ann. **15**, 25–57 (1944); these Rev. **6**, 11] on linkage theory, an explicit formula is given for the distribution of three Mendelian genotypes in the nth generation in terms of the initial distribution and the crossover probabilities. *C. P. Winsor.*

Geiringer, Hilda. On the definition of distance in the theory of the gene. Ann. Math. Statistics **16**, 393–398 (1945). [MF 15474]

The author discusses the relation between her definition of "distance" between genes [same Ann. **15**, 25–57 (1944); these Rev. **6**, 11] and that of J. B. S. Haldane. Comments from the point of view of the biologist are given by I. M. H. Etherington. The differences arise primarily because the author assumes as given the entire set of linkage relations between genes, while the biologists prefer to consider such complete knowledge as only theoretically attainable.

C. P. Winsor (Baltimore, Md.).

Branson, Herman. The flow of a viscous fluid in an elastic tube: a model of the femoral artery. Bull. Math. Biophys. **7**, 181–188 (1945). [MF 14536]

Piaget, Jean. Interprétation probabiliste de la loi de Weber et de celle des centrations relatives. C. R. Séances Soc. Phys. Hist. Nat. Genève **60**, 200–204 (1943). [MF 14219]

Coz Campos, Damisela. On the generalization of the law of radial concentration in the case of a spherical cell. Revista Ci., Lima **47**, 275–300 (1 plate) (1945). (Spanish) [MF 14483]

García has considered the autocatalytic production of a metabolite in a spherical cell with a slowly moving solvent [Revista Ci., Lima **46**, 417–450 (1944) = Actas Acad. Ci. Lima **7**, 317–350 (1944); these Rev. **6**, 240], thus encountering the equation

$$\{d^2f/dr^2+(2/r)df/dr\}r^2/f+C\varphi^2=M$$

governing the radial variation of the concentration of the metabolite. The author essentially retraces García's steps and then shows that Rashevsky's results [Mathematical

Biophysics, University of Chicago Press, 1938] represent a special case. *A. S. Householder* (Washington, D. C.).

Waddell, Agnes H. Curves formed by colonies of micro-organisms growing on a plane surface. Edinburgh Math. Notes no. 35, 14–19 (1945). [MF 14447]

A colony of micro-organisms growing on a plane surface expands uniformly in all directions when there is no interference, the boundary thus being an expanding circle. Two colonies starting from different centers will eventually meet and mutually limit subsequent expansion. If rates of growth are in a constant ratio the locus of contact consists of an arc of a Cartesian oval and symmetric arcs of equiangular spirals. *A. S. Householder* (Washington, D. C.).

GEOMETRY

van Alphen, H. J. Generalization of a theorem of Besicovitsch. Mathematica, Zutphen. B. 10, 144–157 (1942). (Dutch) [MF 15733]

Besicovitch showed that a line segment of length 1 can be turned continuously through 180° in a set of arbitrarily small plane area [Math. Z. 27, 312–320 (1928); simplified proof by Perron, Math. Z. 28, 383–386 (1928)]. The author gives a modified construction showing that the set can be confined to the interior of a circle of radius $2+\gamma$, $\gamma>0$. He shows that the convex cover of Besicovitch's set has area of the order of $1/\epsilon$ as $\epsilon\to 0$, where ϵ is the area of the set.

R. P. Boas, Jr. (Providence, R. I.).

d'Escamard, Vincenzo. Un teorema sul triangolo rettangolo. Boll. Un. Mat. Ital. (2) 4, 200–202 (1942). [MF 16071]

Si dimostra l'inverso del noto teorema: "In ogni triangolo rettangolo il semiperimetro è uguale alla somma del diametro del cerchio circoscritto e del raggio del cerchio iscritto."

Author's summary.

Cattaneo, Paolo. Due teoremi sui triangoli. Boll. Un. Mat. Ital. (2) 5, 35–37 (1943). [MF 16084]

Another proof of the theorem of the paper reviewed above, and of another similar theorem.

Ciamberlini, Corrado. Sulla condizione necessaria e sufficiente affinchè un triangolo sia acutangolo, rettangolo o ottusangolo. Boll. Un. Mat. Ital. (2) 5, 37–41 (1943). [MF 16085]

Narayanamurthy, T. Feuerbach's theorem. Math. Student 13, 43–46 (1945). [MF 15640]

Gerretsen, J. C. H. An analogue of the nine-point circle in the space of n-dimensions. Nederl. Akad. Wetensch., Proc. 48, 535–536 = Indagationes Math. 7, 123–124 (1945). [MF 15805]

Claeys, Marcel. On orthopoles and a generalization. Wis- en Natuurk. Tijdschr. 11, 20–24 (1942). (Dutch) [MF 15581]

Claeys, Marcel. On orthopoles and Wallace lines. Wis- en Natuurk. Tijdschr. 11, 118–121 (1943). (Dutch) [MF 15586]

Bilo, J. Brief contribution to the study of the isopole. Wis- en Natuurk. Tijdschr. 11, 92–103 (1942). (Dutch. French summary) [MF 15584]

Bilo, J. Remarks on two contributions to the study of the orthopole and the isopole. Wis- en Natuurk. Tijdschr. 12, 14–17 (1944). (Dutch) [MF 15587]

Lauwerier, H. A. Some properties of the Wallace lines of a triangle. Nieuw Tijdschr. Wiskunde 28, 248–257 (1941). (Dutch) [MF 15754]

Tummers, J. H. La droite de Wallace comme cas particulier d'un théorème plus général. Mathematica, Zutphen. B. 12, 24–28 (1943). [MF 15726]

Thébault, V. Sphères de Lemoine du tétraèdre. Bull. École Polytech. Bucarest [Bul. Politechn. Bucureşti] 16, 6–18 (1945). [MF 15622]

Gribnau, H. A. Über einige merkwürdige Punkte und Linien des Kreisvierecks. Nieuw Arch. Wiskunde (2) 21, 65–72 (1941). [MF 15714]

Hruša, Karel. On spherical conic sections. Rozhledy Matematicko-Přírodovědecké 21, 78–87, 105–114 (1942). (Czech) [MF 15694]

Van Staey, B. On a mapping of the circles of the plane on the points of space. Wis- en Natuurk. Tijdschr. 11, 80–91 (1942). (Dutch) [MF 15583]

Claeys, A. On a transcendental cone. Wis- en Natuurk. Tijdschr. 11, 73–79 (1942). (Dutch) [MF 15582]

Claeys, A. On a pencil of nodal cubics related by homology. Wis- en Natuurk. Tijdschr. 12, 119–142 (1945). (Dutch) [MF 15590]

Let $ABCP$ be a variable rectangle having two opposite vertices A and C on a fixed line KM, while side AP passes through a fixed point O, and vertex B moves along the line through O perpendicular to KM. The locus of P is a nodal cubic which the author calls a "parabolic leaf," parabolic because it has no asymptote, the line at infinity being an inflectional tangent as we see from the equation $x^3=a(x^2-y^2)$. By means of a homology with center O and axis KM, the author derives the curve $x(x^2-\lambda y^2)=a(x^2-y^2)$, which is still a nodal cubic touching $x=a$ (or KM) and having its node at O, with tangents $x\pm y=0$. But now there is an asymptote $x=a/\lambda$.

When $\lambda=-1$, we have the right strophoid or logocyclic curve, $r\cos\theta=a\cos 2\theta$, which is the inverse of a rectangular hyperbola with respect to a vertex, and also the pedal of a parabola with respect to the foot of the directrix. When $\lambda=-3$, we have the folium of Descartes [see, for example, A. B. Basset, An Elementary Treatise on Cubic and Quartic Curves, Cambridge, 1901, p. 94], which is similar to its own Hessian. When λ is positive there are two oblique asymptotes. The curve consists of two branches when $\lambda>1$, three when $0<\lambda<1$. These remarks are illustrated by diagrams for $\lambda=0$, -3, 3, $\frac{1}{3}$. The author computes the area of the loop and also, for $\lambda<0$, the area between the rest of the curve and its asymptote. Finally, he describes various geometrical constructions. *H. S. M. Coxeter.*

Kuiper, N. H. Lines in R_4. Nieuw Arch. Wiskunde (2) **21**, 124–143 (1941). (Dutch) [MF 15717]

Part A deals with the quintic sixfold in projective 9-space whose points represent the ∞^6 lines in projective 4-space (just as the points on a quadric fourfold in 5-space represent the ∞^4 lines in 3-space). The possible sections by a line, plane, 3-space or 4-space are described exhaustively. Part B applies these results to the theory of lines in 4-space. Here the general linear complex (∞^5 lines) contains a 3-dimensional bundle of ∞^3 lines through an arbitrary point, but contains all the ∞^3 lines through one particular point, called the center (kern) of the complex. The "pencil" of linear complexes determined by two of them has for locus of centers a conic. The common lines of the two complexes (or of the pencil) are a system of ∞^4, containing a flat pencil through an arbitrary point but a bundle through a point on the conic. The "bundle" of linear complexes determined by three of them has for locus of centers a projected Veronese surface V_2^4, whose trisecants are the ∞^3 common lines of the complexes. This system of ∞^3 lines is called a linear congruence, because it contains just one line through an arbitrary point (but a flat pencil through a point on the V_2^4). The ∞^3 linear complexes determined by four of them has for locus of centers Segre's cubic threefold with ten nodes, which is generated by the ∞^2 common lines of the complexes. The common lines of five linear complexes generate an elliptic quintic ruled surface; and the common lines of six are just five associated lines. All these results were anticipated by J. A. Todd [Proc. London Math. Soc. (2) **30**, 513–550 (1930)]. *H. S. M. Coxeter* (Toronto, Ont.).

Castoldi, L. Sopra una conseguenza del teorema del quadrangolo di Desargues che generalizza il teorema dei quadrangoli omologici. Boll. Un. Mat. Ital. (2) **5**, 44–47 (1943). [MF 16087]

Kommerell, Karl. Masskegelschnitt und Trigonometrie. Math. Z. **47**, 738–742 (1942). [MF 15912]

The formulas of non-Euclidean trigonometry may be very simply obtained by using the given triangle as triangle of reference for homogeneous coordinates and expressing its sides and angles in terms of the coefficients in the ordinary and tangential equations for the absolute:

$$\sum\sum f_{ik}x_ix_k=0, \quad \sum\sum \varphi_{ik}u_iu_k=0.$$

This was done in the elliptic case by Baker [Principles of Geometry, vol. 2, Cambridge University Press, 1930, p. 205]. The present author gives the hyperbolic case, which is very similar. [See also Coxeter, Non-Euclidean Geometry, Mathematical Expositions, no. 2, University of Toronto Press, 1942, p. 237; these Rev. **4**, 50.]

H. S. M. Coxeter (Toronto, Ont.).

v. d. Kulk, W. Über den kürzesten Abstand von zwei windschiefen Geraden im elliptischen Raum. Nieuw Arch. Wiskunde (2) **21**, 3–11 (1941). [MF 15708]

Let l and m be two skew lines, not Clifford parallel, in elliptic space; let A and B be variable points on the respective lines. If A and B are related in such a way that the line AB is always perpendicular to m (or always perpendicular to l), then the acute distance AB takes its minimum value when the line AB is one of the two common perpendiculars to l and m and takes its maximum value when it is the other common perpendicular. On the other hand, if A and B are free to vary independently, the acute distance takes its minimum value as before, but the second common perpendicular PQ provides neither a maximum nor a minimum. For, if AB remains perpendicular to m (as above) we have $AB<PQ$; but if A remains at P while B varies from Q, we have $AB=PB>PQ$, since the triangle PBQ is right-angled at Q. This distinction, which was misunderstood by both Darboux [Principes de Géométrie Analytique, Paris, 1917, pp. 308 ff.] and Klein [Vorlesungen über Nicht-Euklidische Geometrie, Berlin, 1928, p. 236, footnote] is now clarified. *H. S. M. Coxeter* (Toronto, Ont.).

Bottema, O. On motions in elliptic space, in which all points describe congruent plane curves. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 25–29 (1944). (Dutch. German, English and French summaries) [MF 14309]

In ordinary space systems of motions with plane congruent trajectories are only possible if the planes of the trajectories are parallel; this is a consequence of Darboux's theorem [Koenigs, Leçons de Cinématique, Paris, 1897, note III, pp. 352–389]. It is shown that in elliptic space analogous systems of motions exist having the property that an arbitrary plane contains a trajectory. If $C(P)$ is the trajectory of the point P and $V(P)$ its plane, then $V(P)$ is the polar plane of P with respect to a linear line complex containing a system of generators of the absolute quadric. If P_1 and P_2 are two points, then the normals in P_1 on $V(P_1)$ and in P_2 on $V(P_2)$ are Clifford-parallels. *D. J. Struik.*

de Mira Fernandes, Aureliano. Distance geometry, 2. Curvatures. Cadernos de Análise Geral, no. 17. Junta de Investigação Matemática, Pôrto, 1945. 15 pp. [paged 17–31] (Portuguese) [MF 15591]

[Part 1 appeared in the same Cadernos, no. 13 (1945); these Rev. **7**, 24.] This article is essentially a reproduction of pages 363–369 of an address by Menger [Enseignement Math. **35**, 348–372 (1936)]. *L. M. Blumenthal.*

de Mira Fernandes, Aureliano. Distance geometry, 3. Arc length. Cadernos de Análise Geral, no. 18. Junta de Investigação Matemática, Pôrto, 1946. 18 pp. [paged 33–50] (Portuguese) [MF 15592]

This article is a résumé of portions of Menger's Metrische Methoden in der Variationsrechnung [Ergebnisse eines Mathematischen Kolloquiums **8**, 1–32 (1937)].

L. M. Blumenthal (Columbia, Mo.).

Fabricius-Bjerre, Fr. Über lineare und quadratische Abbildungen des Raumes auf eine Ebene. Danske Vid. Selsk. Math.-Fys. Medd. **22**, no. 5, 40 pp. (1945). [MF 15408]

In the first chapter the author considers oblique axonometric mappings of the space and generalizes slightly the results of a paper by L. Eckhart [Akad. Wiss. Wien. S.-B. IIa **146**, 51–56 (1937)]. The second chapter deals with linear mappings of the space on an image plane. A linear mapping of the space may be composed of a central projection of the space on a projection plane and a subsequent projective transformation of the projection plane into the image plane [E. Müller and E. Kruppa, Die Linearen Abbildungen, Deuticke, Wien, 1923, p. 166]. Two linear mappings (T_1) and (T_2) on the same image plane and two points U_1 and U_2 in this plane are given. A point in space has image points P_1 and P_2 in the mappings (T_1) and (T_2), respectively. The rays U_1P_1 and U_2P_2 are called corresponding rays of the pencils through U_1 and U_2. The point of intersection of two

corresponding rays determines a third mapping (T_3). Let O_i be the center of projection in the mapping (T_i). The point U_i is the image of a line u_i through O_i in (T_i). The line joining the points U_1 and U_2 is the image of a plane σ_i through U_i in (T_i), $i=1, 2$. The mutual position of the lines u_1 and u_2 and the planes σ_1 and σ_2 determines the character of the derived mapping (T_3). Generally, this new mapping is a quadratic mapping. A detailed discussion of this mapping is carried out and particular cases and singular transformations are considered. A typical result is the following theorem: if the lines u_1 and u_2 are coplanar and if their plane is represented in both given mappings by the line $\overline{U_2U_1}$ the derived mapping is a linear mapping. The influence of the choice of the points U_1 and U_2 upon the derived mapping is also discussed. *E. Lukacs* (Cincinnati, Ohio).

Convex Domains, Integral Geometry

Herglotz, G. Über die Starrheit der Eiflächen. Abh. Math. Sem. Hansischen Univ. **15**, 127–129 (1943). [MF 15833]

An elegant new proof for the rigidity of ovals in E^3. The main step is the proof of the formula

$$\int_{F_1}\begin{vmatrix} L_1-L_2 & M_1-M_2 \\ M_1-M_2 & N_1-N_2 \end{vmatrix}(ax+by+cz)do=\int_{F_2}H_2do-\int_{F_1}H_1do,$$

where it is assumed that the first fundamental forms of the ovals F_1, F_2 are identical. Here L_i, M_i, N_i are the coefficients of the second fundamental form of F_i, H_i is the mean curvature of F_i, and a, b, c are the direction cosines of the normal to F_1. *H. Busemann* (Northampton, Mass.).

Blaschke, Wilhelm. Sulla rigidità degli ovaloidi. Boll. Un. Mat. Ital. (2) **5**, 107–114 (1943). [MF 16095]

An expository lecture.

Gericke, H. Stützbare Bereiche in komplexer Fourier-Darstellung. Deutsche Math. **5**, 279–299 (1940). [MF 14334]

A supportable region (stützbarer Bereich) in the plane is one bounded by a closed curve whose normals take every direction exactly once. Such regions can be described by means of the Fourier series

$$A(\varphi)=\sum_{k=-\infty}^{+\infty}\alpha_k e^{ik\varphi}, \qquad \alpha_k=\bar{\alpha}_{-k},$$

which represents their function of support, in the same way as convex regions. The present paper contains additions to two papers by Görtler [Deutsche Math. **2**, 454–466 (1937); **3**, 189–200 (1938)], which deal with the representation of general supportable regions as sums of special ones which form a "basis." The simplest basis consists of the hypocycloids corresponding to

$$A(\varphi)=\alpha_k e^{ik\varphi}+\bar{\alpha}_k e^{-ik\varphi}. \tag{1}$$

The author is mostly concerned with the relation of a supportable region to its "associated" regions, which are defined by

$$\alpha_k^*=\begin{cases}\alpha_k e^{i\omega}, & k>0,\\ \alpha_0\cos\omega, & k=0,\\ \alpha_k e^{-i\omega}, & k<0,\end{cases}$$

with some real ω. It is shown that associated regions have the same isoperimetric defects. The author proves that the only regions for which all associated regions can be obtained by rotation of the original curve are the hypocycloids of the form (1). He also determines all regions for which there exists one associated region obtainable by rotation.

To each $A(\varphi)$ there also corresponds the minimal surface defined by the analytic function $f(z)=\sum_k \alpha_k z^k$ in the Weierstrass representation. Associated curves then correspond to associated minimal surfaces in the sense of Bonnet [see Darboux, Leçons sur la Théorie Générale des Surfaces, livre III, chap. V, Gauthier-Villars, Paris, 1894]. The hypocycloids (1) correspond to certain algebraic minimal surfaces developable on surfaces of revolution. *F. John.*

Hadwiger, H. Kleine Studien zum Tangentialpolyeder. Elemente der Math. **1**, 11–12 (1946). [MF 15691]

Let V and F denote volume and area of a convex polyhedron Π in E^3 all of whose faces are tangent to one sphere (Inkugel). Put $K=\sum s_i\cot(\varphi_i/2)$, where s_i is the length of an edge of Π and φ_i is the angle formed by the two faces of Π contiguous at s_i. The following analogues of known inequalities of Minkowski are proved by elementary means: $K^2-4\pi F>0$, $F^2-3KV\geqq 0$. *H. Busemann.*

Blumenthal, O., and Wolff, J. The isoperimetric problem. Mathematica, Zutphen. B. **11**, 12–26 (1942). (Dutch) [MF 15732]

Expository article.

Algebraic Geometry

Nakayama, Tadasi. A remark on the "zugeordnete" form, function field and tangent space of an algebraic variety over an imperfect modular field. Proc. Phys.-Math. Soc. Japan (3) **23**, 883–891 (1941). [MF 15011]

Let $\{\xi_1, \xi_2, \cdots, \xi_n\}=\{\xi\}$ be the general point of an irreducible algebraic variety M, of dimension d, over a given ground field K, and let $\{u_0^{(j)}, u_1^{(j)}, \cdots, u_n^{(j)}\}=\{u^{(j)}\}$, $j=0, 1, \cdots, d$, be $d+1$ sets of indeterminates $u_i^{(j)}$. Let

$$F(\{u^{(0)}\}, \{u^{(1)}\}, \cdots, \{u^{(d)}\}) = \rho(\{u^{(1)}\}, \{u^{(2)}\}, \cdots, \{u^{(d)}\})\cdot\prod_{\alpha=1}^{g}F_\alpha$$

be the associated form of M in the sense of van der Waerden, where $F_\alpha=u_0^{(0)}+\sum_{i=1}^n u_i^{(0)}\xi_i^{(\alpha)}$ and $\{\xi^{(1)}\}, \{\xi^{(2)}\}, \cdots, \{\xi^{(g)}\}$ are the intersections of M with the $(n-d)$-space

$$u_0^{(1)}+\sum_{i=1}^n u_i^{(1)}x_i=u_0^{(2)}+\sum_{i=1}^n u_i^{(2)}x_i=\cdots=u_0^{(d)}+\sum_{i=1}^n u_i^{(d)}x_i=0.$$

The following result is proved: the g factors F_α are distinct if and only if the set $\{\xi\}$ contains a separating transcendence basis of Σ/K, where $\Sigma=K(\{\xi\})$. The equivalence of these two conditions is proved by showing that either one is equivalent to the following: if $\{f_1(x), f_2(x), \cdots, f_N(x)\}$ is a basis of the prime ideal of the variety M in the polynomial ring $K[x_1, x_2, \cdots, x_n]$, then the Jacobian matrix $\|\partial f_s/\partial x_t\|_{s,t}$ is of rank $n-d$ (in the terminology of the author: M has a tangent space at its generic point $\{\xi\}$). It may be pointed out that the author's main result can also be expressed in the following form: the set $\{\zeta_1, \zeta_2, \cdots, \zeta_g\}$, $\zeta_j=\sum_{i=1}^n u_i^{(j)}\xi_i$, is a separating transcendence base of $\Sigma(\{u_i^{(j)}\})/K(\{u_i^{(j)}\})$, $i=1, 2, \cdots, n$; $j=1, 2, \cdots, d$, if and only if the set $\{\xi\}$ contains a separating transcendence basis of Σ/K. In this form it appears clearly that this result is equivalent to the follow-

ing theorem of MacLane [Duke Math. J. 5, 372–393 (1939); see lemma 2, p. 380, and theorem 15, p. 384]: if a field Δ is a finite separably generated extension of a field k, then any set of generators of Δ/k contains a separating transcendence basis for Δ/k. [There is a minor inaccuracy in the proof of lemma 1 on p. 886: the statement at the bottom of the page that "the ideal generated by the polynomials (8) in $K[x_1, x_2, \cdots, x_n]$ is prime" is not correct (moreover, read x_n instead of x_d). As a consequence, in the relation $f(x)=g_{d+1}(x)f_{d+1}(x)+\cdots+g_n(x)f_n(x)$ one should replace $f(x)$ by $h(x_1, x_2, \cdots, x_d)f(x)$, where h is a suitable polynomial in $x_1, x_2, \cdots, x_d$. The factor h does not interfere with the rest of the proof.] *O. Zariski* (Urbana, Ill.).

Châtelet, François. Variations sur un thème de H. Poincaré. Ann. Sci. École Norm. Sup. (3) 61, 249–300 (1944). [MF 14651]

It is known (R being the rational number field) that (I) a unicursal curve in R contains points in R if and only if it is birationally equivalent in R to a line and (2) a unicursal curve in R is equivalent in R either to a line or a conic in R [Poincaré, J. Math. Pures Appl. (5) 7, 161–233 (1901); M. Noether, Math. Ann. 23, 311–358 (1884); Hilbert and Hurwitz, Acta Math. 14, 217–224 (1891)]. The author gives new and suggestive proofs of these results, which he is able to generalise.

Let Φ be an arbitrary field and F an r-dimensional "Brauer variety" in Φ, that is, let F be defined by equations with coefficients in Φ and let there be a Poincaré transformation (birational with only isolated critical points) $\mathfrak{T}$ of a projective space E_r into F, the coefficients of $\mathfrak{T}$ being in some algebraic extension k of Φ. Every Poincaré transformation of F into itself in k is of the form $\mathfrak{T}^{-1}\mathfrak{C}\mathfrak{T}$, where $\mathfrak{C}$ is a nondegenerate homography in k of E_r, and this lies in Φ if and only if $\mathfrak{C}$ satisfies certain equations called the "system of equivalence" of F. The aggregate of degenerate and nondegenerate solutions $\mathfrak{C}$ of these equations forms a normal simple algebra over Φ with k as a splitting field. The degenerate solutions give rise to "normal subvarieties" of F (which are themselves Brauer varieties in Φ) and solutions of rank one correspond to points of F in Φ. The conditions for equivalence (defined in terms of Poincaré transformations) between Brauer varieties or their normal subvarieties reduce to simple relations between the corresponding algebras. Among other results, the extension to Brauer varieties of the first theorem above is derived by an application of the Wedderburn structure theorem. *D. B. Scott.*

Wiman, A. Über rationale Punkte auf Kurven $y^2=x(x^2-c^3)$. Acta Math. 77, 281–320 (1945). [MF 14635]

In a previous paper the author has studied the rank of the cubics $y^2=x(x+a)(x+b)$ [Acta Math. 76, 225–251 (1945); these Rev. 7, 70]. Here similar methods are applied to the curve $y^2=x(x^3-c^3)$, where c is a positive integer free from square factors. It is proved that the rank r of this curve is not greater than $2n+2$, where n denotes the number of odd prime factors of c. A lower upper bound of r can sometimes be found, and the exact value of r may be obtained in certain cases by using the Abelian group associated with the rational points of the curve; thus it is shown, for example, that $r=4, 5$ if $c=2\cdot3\cdot5\cdot11$, $2\cdot3\cdot11\cdot19$, respectively.

The case $n=1$ is studied in detail. If c is a prime number of the form $8k+3$, then $r=2$; if $c=8k+5$ or $8k+7$, then $2\leqq r\leqq3$; if $c=8k+1$, then $2\leqq r\leqq4$. For example, $r=3$ for $c=5$, 13, 29, 37, 61 and $r=4$ for $c=41$, 137, 761. If $c=2p$ is the double of a prime number of the form $p=8k+5$ or $16k+9$, then $r=2$; if $p=8k+3$ or $8k+7$, then $2\leqq r\leqq3$; if $p=16k+1$, then $2\leqq r\leqq4$. For example, $r=3$ for $c=2\cdot31$, $2\cdot47$, $2\cdot79$, and $r=4$ for $c=2\cdot17$, $2\cdot97$, $2\cdot113$. In order that $r>4$ it is necessary that $n\geqq2$, but this condition is, of course, far from being sufficient. It is shown that $r\geqq5$ if $c=2\cdot3\cdot5\cdot7\cdot17\cdot43$, $2\cdot3\cdot7\cdot11\cdot19\cdot67$, $2\cdot3\cdot11\cdot19$; $r=5$ if $c=2\cdot3\cdot7\cdot17\cdot31$, $2\cdot3\cdot73\cdot97$, $2\cdot3\cdot97\cdot193$, and a method is given for deriving other curves (probably infinite in number) having $r\geqq5$. Finally, it is proved that $r\geqq6$ if $c=2\cdot3\cdot7\cdot17\cdot41$. *B. Segre* (Manchester).

Hodge, W. V. D. On multiple curves. III. Proc. Cambridge Philos. Soc. 42, 11–14 (1946). [MF 14406]

[Parts I and II appeared in the same Proc. 41, 111–117, 117–126 (1945); these Rev. 7, 27.] If k is the field of complex numbers, $k(C)$ is the function field of C, and $k(\Gamma)$ the function field of a normal multiple Γ of C, then $k(\Gamma)$ is a normal extension of $k(C)$, whose Galois group is isomorphic with the group $\mathfrak{G}$ of birational transformations of Γ into itself [cf. the reviews of parts I and II]. A multiple Γ of any curve C can be found having an arbitrary group $\mathfrak{G}$, as the author proves by constructing $k(\Gamma)$ given $\mathfrak{G}$ and $k(C)$. *D. B. Scott* (London).

Eger, Max. Les systèmes canoniques d'une variété algébrique à plusieurs dimensions. Ann. Sci. École Norm. Sup. (3) 60, 143–172 (1943). [MF 14640]

In the present treatment of canonical systems on an algebraic variety V_n, these systems are first introduced (at least up to a certain dimension) on irregular varieties. Let q be the irregularity of V_n and let $u_0, \cdots, u_\lambda$ be $\lambda+1$ functionally independent differential Picard forms of the first kind, $\lambda+1\leqq q$. The $\lambda+1$ transcendental pencils $u_i=0$ determine on V_n an algebraic subvariety $\{u_0, \cdots, u_\lambda\}$ of dimension λ, their contact variety, defined as follows: at each point P of the contact variety there exists a tangent linear $[n-\lambda]$ common to the hypersurface of each pencil passing through P. Analytically, the variety $\{u_0, \cdots, u_\lambda\}$ is given by the vanishing of the Jacobian $\|\partial u_j/\partial(dt_k)\|$, where the t_k are local uniformizing parameters on V_n. It is proved that $\{u_0, \cdots, u_\lambda\}$ varies in a system of equivalence (in the sense of Severi), and this system $K_\lambda=K_\lambda(V)$ is defined as the canonical system of dimension λ.

Contact varieties of sets of linear systems of hypersurfaces are definable geometrically on any V_n. The contact problem of determining these varieties is solved by the author in terms of the operator K_λ whenever this operator is defined. For instance, it is shown that if $\varphi_0, \cdots, \varphi_\lambda$ are rational functions, then the contact variety $\{\varphi_0, \cdots, \varphi_\lambda\}$ of the $\lambda+1$ pencils $\varphi_i=$constant is given by the symbolic expression $K_\lambda\prod_i(1+\varphi_i)^2$, where products $\varphi_i\varphi_j$ denote complete intersections and where V_n itself is treated as a unit element 1. Now if V_n is an arbitrary variety of dimension n and if it is assumed that the operator K_λ has already been defined for every variety of dimension less than n, then the relation $\{\varphi_0, \cdots, \varphi_\lambda\}\equiv K_\lambda\prod_i(1+\varphi_i)^2$ can be used to define a system of equivalence $K_\lambda(V)$ on V_n (possibly only a virtual system), and since this system is proved to be independent of the rational functions $\varphi_0, \cdots, \varphi_\lambda$, the author obtains in this fashion a general definition of canonical systems.

The paper also deals with adjoint systems of linear systems of hypersurfaces F on V_n. An interesting symbolic expression is derived for the adjoint system A_λ^F of dimension λ and the canonical systems of complete intersections on V_n are then expressed in terms of adjoint systems. This enables the author to write $K_\lambda(V_n)$ in a form which involves

only geometric entities related to V_n itself. Some applications are given, such as canonical systems in a projective space and the Jacobian of a linear system of hypersurfaces. *O. Zariski* (Urbana, Ill.).

de Franchis, Michele. I sistemi canonici e pluricanonici e le forme algebrico-differenziali di prima specie. Ann. Mat. Pura Appl. (4) **19**, 243–249 (1940). [MF 15086]

The author indicates a simple way of obtaining the canonical systems of equivalence on an algebraic variety by considering certain differential forms, thus extending a method of obtaining the canonical series on the curve [M. de Franchis, Rend. Circ. Mat. Palermo **34**, 165–168 (1912)]. Thus, for instance, if $du\,dv=\sum a_{ij}dx_i dx_j$ is a double differential of the first kind attached to an algebraic variety V then the set of points on V for which all the functions a_{ij} vanish is (in general) a canonical curve on V, and the notion can clearly be extended. *J. A. Todd.*

Longhi, Ambrogio. Sulla intersezione di due o più varietà algebriche. Comment. Math. Helv. **18**, 45–51 (1945).

Generalizing the concept of section genus (first used by G. Fano as a characteristic of a line congruence in S_3 and for which the name was suggested by C. Segre), the author defines the section genus of an algebraic variety V_ρ of dimension $\rho>1$ in S_r to be the genus of the complete intersection of V_ρ and a linear space $S_{r-\rho+1}$. The following general theorem is proved. In S_r, $r\geqq 3$, ν varieties $V_{\rho_i}^{n_i}$, generically situated, of respective dimensions $\rho_i\leqq r-1$, orders n_i, and section genera p_i, $i=1,\cdots,\nu$, each possessing any singularities, intersect in a variety V_k^n of dimension $k>0$ and order $n=\prod n_i$, whose section genus p is

$$p=1+n(\nu-1)+n\sum(p_i-1)/n_i$$

when $k>1$. If $k=1$, V_k^n is a curve whose genus is the value of p. Particular cases of this theorem are given and also conditions for the reducibility of V_k^n. *T. R. Hollcroft.*

Zappa, Guido. Sulle involuzioni di una varietà algebrica ad r dimensioni dotate di al più ∞^{r-2} punti di coincidenza. Boll. Un. Mat. Ital. (2) **5**, 10–13 (1943). [MF 16079]

Godeaux, Lucien. Sur les surfaces du cinquième ordre circonscrites à un hexaèdre complet. Bull. Soc. Math. France **73**, 27–42 (1945). [MF 14173]

The author proves that a quintic surface in S_3 which contains the fifteen edges of a complete hexahedron is the image of three involutions, of orders 2, 3 and 6, on algebraic surfaces whose characters are determined. The paper is an application of the author's theory of cyclic involutions with a finite number of united points. *J. A. Todd.*

Godeaux, Lucien. Une propriété des variétés de Veronese. Nederl. Akad. Wetensch., Proc. **44**, 1058–1061 (1941). [MF 15766]

Franchetta, Alfredo. Sulle superficie le cui curve canoniche contengono una g_3^1. Boll. Un. Mat. Ital. (2) **4**, 238–243 (1942). [MF 16075]

Lesieur, Léonce. Sur la représentation rationnelle d'une hyperbiquadratique. Bull. Soc. Math. France **73**, 43–54 (1945). [MF 14174]

Observations on the representation of the quartic variety common to two quadrics in S_n $(n>3)$ on an S_{n-2}. Apart from some explicit equations the author has little to add to the work of Rosati [Ann. Mat. Pura Appl. (3) **1**, 25–37 (1898)], whose paper he refers to. *J. A. Todd.*

Edge, W. L. A plane quartic curve with twelve undulations. Edinburgh Math. Notes no. **35**, 10–13 (1945). [MF 14446]

Among the curves of the pencil

$$x^4+y^4+z^4+\lambda(y^2z^2+z^2x^2+x^2y^2)=0$$

(which are all invariant under an octahedral group of collineations), there exists one, namely Dyck's curve, given by $\lambda=0$, whose inflections are known to coincide in pairs at twelve undulations. The author shows that another curve of the pencil, namely that for which $\lambda=3$, has the same property; his subsequent discussion of this curve includes a simple geometrical construction for the configuration of the twelve undulations. *J. G. Semple* (London).

Galafassi, Vittorio Emanuele. Modelli minimi di curve algebriche reali col massimo numero di circuiti dispari non secantisi, sui singoli tipi di superficie cubiche reali. Boll. Un. Mat. Ital. (2) **4**, 166–171 (1942). [MF 16067]

Rossier, Paul. Sur les points paraboliques d'une courbe algébrique. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 40–41 (1942). [MF 14195]

If the polar conic of a point of an algebraic curve is a parabola, the point is said by the author to be parabolic [same C. R. **58**, 207–209 (1941); these Rev. **3**, 299]. Here it is shown that if a nondegenerate algebraic curve of degree n has $2n(n-2)+1$ parabolic points all the points of the curve are parabolic. The proposition is also generalized by projection. *N. A. Court* (Norman, Okla.).

Rossier, Paul. Condition d'osculation de la première polaire relative à une courbe algébrique plane. C. R. Séances Soc. Phys. Hist. Nat. Genève **62**, 61–62 (1945). [MF 14559]

Andréani, J. Sur une définition des cycliques planes à point double. Ann. Fac. Sci. Univ. Toulouse (4) **6**, 7–14 (1943). [MF 15171]

If two points P, P' of the plane are associated by the conditions (i) they are equidistant from a fixed point A, (ii) their join passes through a fixed point O, the resulting correspondence between P and P' carries lines into circular cubics nodal at O and circles into bicircular quartics nodal at O. Various elementary curves are exhibited as particular cases. *J. G. Semple* (London).

Apéry, Roger. Sur les défauts des séries découpées par les formes d'ordre k sur deux courbes complémentaires. C. R. Acad. Sci. Paris **221**, 436–438 (1945). [MF 14685]

This note discusses the following lemma. The postulation of a variety V having ρ connected components on which the forms of order k cut a series g of deficiency δ and dimension $r-\delta$ is equal to $r+\delta-\delta^*$, where $0\leqq\delta^*-\delta\leqq\rho-1$. The question is whether certain alternative series lead to equivalent postulation [Apéry, same C. R. **220**, 234–236 (1945); these Rev. **7**, 170]. [Enriques and Chisini, Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche, vol. 3, Bologna, 1924, p. 537, give the formula incorrectly.] *V. Snyder* (Ithaca, N. Y.).

Gambier, Bertrand. Sur un principe de géométrie énumérative basé sur le théorème de d'Alembert. Bull. Sci. Math. (2) 65, 264–279 (1941). [MF 13259]

The author considers the compatibility of a set of algebraic equations (whose solutions would contain m parameters) defining certain properties of a system of varieties V_m of dimension m. His method consists of a geometric study of relations between V_m and a variety W_m representing the set of conditions defined by the given set of equations. He illustrates the principle involved by several examples. *T. R. Hollcroft* (Aurora, N. Y.).

Differential Geometry

Kasner, Edward, and De Cicco, John. Converse theory of gnomic and equiareal perspectivities. Proc. Nat. Acad. Sci. U. S. A. 32, 16–19 (1946). [MF 15313]

Let $z=f(x, y)$ be the equation of a surface Σ and let $p\equiv\partial z/\partial x$, $q\equiv\partial z/\partial y$, $r\equiv\partial^2 z/\partial x^2$, $s\equiv\partial^2 z/\partial x\partial y$, $t\equiv\partial^2 z/\partial y^2$. If the origin is taken as the center of perspectivity, then the perspective map of Σ upon the plane $\Pi: z=c\neq 0$ is given by $x'=cx/z$, $y'=cy/z$, $z'=c$.

It is found that the geodesics which are projected into straight lines satisfy either (1) $r+2s\,dy/dx+t(dy/dx)^2=0$ or

$$(2)\quad \{(1+q^2)(pz+x)-pq(qz+y)\}dy/dx = (1+p^2)(qz+y)-pq(pz+x).$$

A discussion of (1) and (2) yields the following classification of surfaces according to the number of geodesics projected into straight lines (that is, mapped gnomically) by a perspectivity: (i) at most ∞^1 (nonruled surfaces), (ii) always ∞^1 but at most $2\infty^1$ (ruled surfaces but not quadrics), (iii) always $2\infty^1$ but at most $3\infty^1$ (quadrics but not gnomic projections of spheres) and (iv) all ∞^2 (gnomic projections of spheres and the limiting cases of planes).

It is found that all surfaces Σ for which area-preserving perspectivities exist must satisfy

$$(3)\quad c^4(z-px-qy)^2=z^4(1+p^2+q^2).$$

The complete integrals of (3) are cylinders with elements parallel to, and symmetric with respect to, a line through the origin; all elements are parallel to the plane Π. The directrices of the cylinders are given parametrically in terms of a Chebyshev integral which (by a theorem of Liouville) cannot be expressed in terms of the elementary functions. *M. O. Reade* (Lafayette, Ind.).

Kasner, Edward, and De Cicco, John. General theory of scale curves. Amer. J. Math. 68, 66–76 (1946). [MF 15490]

Let (x, y) denote general curvilinear coordinates of a point on a surface Σ and Cartesian coordinates on a plane Π. The square of the linear element on Σ is

$$dS^2=E(x, y)dx^2+2F(x, y)dxdy+G(x, y)dy^2,$$

$H\equiv\{EG-F^2\}^{\frac{1}{2}}>0$, while the square of the linear element on Π is $ds^2=dx^2+dy^2$. A cartogram is any mapping of Σ on Π, with scale function defined by $\sigma\equiv ds/dS$. In general, σ depends on (x, y) and $y'\equiv dy/dx$. The curve $\sigma=$constant is called a scale curve; for nonconformal cartograms there are ∞^2 scale curves. In this paper, the authors study the geometry of systems of scale curves in Π for nonconformal cartograms.

Since the scale curves are defined by $ds/dS=$constant, they satisfy the differential equation

$$(1)\quad y''=\frac{(1+y'^2)\{E_x+y'(E_y+2F_x)+y'^2(2F_y+G_x)+y'^3G_y\}}{2\{-F+y'(E-G)+y'^2F\}}.$$

Since (1) is of special algebraic form in the first derivative, not every system of ∞^2 curves in Π can represent the scales of a nonconformal cartogram. It follows from (1) that there are essentially ∞^2 surfaces which possess the same family of ∞^2 scale curves. Moreover, if the scale curves given by (1) are all straight lines, then

$$E_x=0,\quad E_y+2F_x=0,\quad 2F_x+G_x=0,\quad G_y=0;$$

hence it follows that the class of surfaces Σ for which there exists a map upon Π such that the scale curves coincide with the totality of ∞^2 straight lines in Π as linear element satisfy

$$(2)\quad dS^2=a_0(ydx-xdy)^2+2(a_1dx+b_1dy)(ydx-xdy) +a_2dx^2+2c_2dxdy+b_2dy^2.$$

It follows that, in the class of surfaces (2) for which $a_0^2+a_1^2+b_1^2\neq 0$, the surfaces exist on one side of the conic section $H=0$; moreover, the Gaussian curvature is constant along certain conic sections associated with the conic $H=0$. Finally, to a given scale σ there correspond ∞^1 straight scales which are all tangent to a conic confocal with $H=0$. *M. O. Reade* (Lafayette, Ind.).

De Cicco, John. Conformal maps with isothermal systems of scale curves. Amer. J. Math. 68, 137–146 (1946). [MF 15496]

The author deals with isothermal systems of curves which represent scale curves [see the preceding review] of a conformal map of a surface Σ upon a plane, where Σ is restricted to be applicable upon a surface of revolution. Such a surface Σ may be (i) sphere-like, if it has constant nonzero Gaussian (total) curvature, (ii) vase-like, if it is applicable upon a surface of revolution of variable Gaussian curvature, or (iii) a developable; the author obtains differential conditions in order that the surface Σ be of one of these three mutually exclusive types.

The main problem solved in this paper is the determination of the possible forms of the linear-elements of all vase-like surfaces Σ for which there exist conformal maps upon a plane with isothermal systems of scale curves which are neither parallel straight lines nor concentric circles. It is found by direct methods that there are essentially seven basic forms in the real domain, or six in the complex domain; a typical form is

$$ds^2=\{(au+b)(cv+d)\}^{n-2}(u+v)^{-n}dudv,\quad n\neq 0, 2,$$

where $u=x+iy$, $v=x-iy$, and a, b, c, d, n are constants. *M. O. Reade* (Lafayette, Ind.).

Wenedikoff, M. Formeln für die winkeltreue schiefachsige Zylinderprojektion. Schweiz. Z. Vermessgswes. Kulturtech. 40, 64–69 (1942). [MF 15570]

Tsuji, Masatsugu. On a theorem of F. and M. Riesz. Proc. Imp. Acad. Tokyo 18, 172–175 (1942). [MF 14750]

Let the functions $x_j=x_j(u, v)$, $j=1, \cdots, n$, map $D: u^2+v^2<1$ conformally on a minimal surface S, bounded by a rectifiable curve Γ, in Euclidean n-space. Then there is induced a topological mapping of $C: u^2+v^2=1$ on Γ. A theorem of F. and M. Riesz concerning analytic functions of a complex variable is generalized to show that to each

null set on C there corresponds a null set on Γ, and conversely. It is shown further that the mapping remains conformal at almost all points of C. *E. F. Beckenbach.*

Egerváry, E. Über die Schmiegungskugeln der Kurven des n-dimensionalen Euklidischen Raumes. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 775–786 (1940). (Hungarian. German summary) [MF 15559]

In vorliegender Arbeit wird eine von H. A. Schwarz [Ann. Mat. Pura Appl. (2) **10**, 129–137 (1881)] herrührende Grenzwertsatz auf Funktionen von zwei Variabeln erweitert und mit dessen Hilfe der folgende Ausdruck für den Radius R_{k-1} der $(k-1)$-dimensionalen Schmiegungskugel abgeleitet:

$$R_{k-1}^2=-\begin{vmatrix} & q' & \cdots & q^{(k)} \\ q' & X_{11} & \cdots & X_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ q^{(k)} & X_{k1} & \cdots & X_{kk} \end{vmatrix} : \begin{vmatrix} X_{11} & \cdots & X_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ X_{k1} & \cdots & X_{kk} \end{vmatrix},$$

wo $X_{ij}=\sum_{\nu=1}^{n} x_\nu^{(i)}x_\nu^{(j)}$, $k=2, 3, \cdots, n$, und

$$q^{(k)}=\frac{1}{2}\frac{\partial^k}{\partial s^k}\sum_{\nu=1}^{n}[x_\nu(s)-x_\nu(t)]^2_{t=s}.$$

Die Elimination der Derivierten der Koordinaten ergibt folgenden Zusammenhang zwischen den Krümmungen $1/\rho_k$ und den Schmiegungskugelradien:

$$R_k^2(dR_k/ds)^2=(R_{k+1}^2-R_k^2)(R_k^2-R_{k-1}^2)/\rho_{k+1}^2,$$

$k=1, 2, \cdots, n-2$; $R_0=0$, $R_1=\rho_1$.

Diese Gleichung kann auch direkt auf Grund von elementaren stereometrischen Sätzen durch Grenzübergang abgeleitet werden. Für $k=n-1$ tritt an die Stelle dieser Gleichung

$$R_{n-1}^2(dR_{n-1}/ds)^2=(R_{n-1}^2-R_{n-2}^2)(d\sigma/ds)^2,$$

wo $d\sigma$ die Entfernung der Mittelpunkten von zwei benachbarten Schmiegungshyperkugeln bedeutet.

Die mit geeigneten Vorzeichen versehene Grössen $\Delta_k=(R_k-R_{k-1}^2)^{\frac{1}{2}}$ $(k=1, 2, \cdots, n-1)$ genügen Relationen, welche eine bemerkenswerte Analogie zu den verallgemeinerten Frenetschen Gleichungen aufweisen. Werden $\Delta_1, \Delta_2, \cdots$ auf Grund von diesen Relationen, als Funktion der Krümmungen $1/\rho_k$ und ihrer Derivierten ausgedrückt, so wird $R_k^2=\Delta_1^2+\cdots+\Delta_k^2$. *From the author's summary.*

Egerváry, E. Über die Kurven des n-dimensionalen Euklidischen Raumes. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 787–797 (1940). (Hungarian. German summary) [MF 15560]

Ist eine Kurve im n-dimensionalen Euklidischen Raum durch die n-mal stetig differenziierbaren Funktionen $x_1=x_1(s), \cdots, x_n=x_n(s)$, der Bogenlänge s bestimmt und sind für $s=s_0$ die Gramsche Determinanten

$$G_k=\begin{vmatrix} X_{11} & \cdots & X_{1k} \\ \cdots & \cdots & \cdots \\ X_{k1} & \cdots & X_{kk} \end{vmatrix}, \quad X_{ij}=\sum_{\nu=1}^{n} x_\nu^{(i)}(s)x_\nu^{(j)}(s),$$

$k=1, \cdots, n$, alle von 0 verschieden, so gelten folgende Sätze.

(I) Das Verhältnis des Winkels η_k der zu den Kurvenpunkten $P(s_1)$ und $P(s_2)$ gehörenden k-dimensionalen Schmiegungshyperebenen zur Bogenlänge s_2-s_1 hat für s_1, $s_2\to s_0$ den Grenzwert

(*) $d\eta_k/ds=(G_{k+1}G_{k-1})^{\frac{1}{2}}/G_k$, $\quad G_0=G_1=1$, $1\leqq k\leqq n-1$.

(II) Die Funktion

$$\frac{(k+1)^2}{k}\,\frac{V(P_1P_2\cdots P_{k+1}P_{k+2})\,V(P_2\cdots P_{k+1})}{\overline{P_1P_{k+2}}\,V(P_1P_2\cdots P_{k+1})\,V(P_2\cdots P_{k+1}P_{k+2})}$$

der Kantenlängen eines Sehnensimplex $[P(s_1)P(s_2)\cdots P(s_{k+2})]$, wo $V(P_1\cdots P_k)$ den $k-1$ dimensionalen Inhalt von $[P(s_1)\cdots P(s_k)]$ bedeutet, hat für $s_\nu\to s_0$ ebenfalls den Grenzwert (*).

Wird demgemäss die k-te Krümmung $1/\rho_k$ als das Verhältnis des Kontingenzwinkels der k-dimensionalen Schmiegungshyperebene zum Bogenelement definiert, so stimmt sie einerseits als Funktion der Derivierten mit den von W. Blaschke auf formalem Wege definierten Krümmungen [Math. Z. **6**, 94–99 (1920)] überein, während andererseits Satz (II) eine Definition der Krümmungen auch in Räumen ermöglicht, welche bloss durch Einführung des Distanzbegriffes metrisiert sind.

Wird diese Krümmungsdefinition zugrundegelegt, so erhalten sowohl die verallgemeinerte Frenetsche Formeln, als auch die auf das begleitende n-Bein bezügliche Koordinatenentwickelungen eine einfache und symmetrische Form. *From the author's summary.*

Bompiani, Enrico. Sul birapporto di quattro punti di una curva. Boll. Un. Mat. Ital. (2) **4**, 84–86 (1942). [MF 16059]

Si dimostra che un procedimento fornito recentemente dal Marletta [Boll. Accad. Gioenia Sci. Nat. Catania (3) **17**, 1–3 (1941)] per definire un invariante proiettivo di quattro punti di una curva, non risolve affatto il problema di costruire il parametro proiettivo che normalizza un'equazione differenziale lineare omogenea. *Author's summary.*

van der Woude, W. On the method of the moving trihedral. Application to the differential geometry of space curves. Mathematica, Zutphen. B. **11**, 1–7 (1942). (Dutch) [MF 15730]

van Gruting, C. J. The equation of the curves of striction of a family of curves lying on a surface in elliptic space. Nieuw Arch. Wiskunde (2) **21**, 101–110 (1941). (Dutch) [MF 15716]

Blaschke, W. Una proprietà caratteristica differenziale delle quadriche. Boll. Un. Mat. Ital. (2) **5**, 88–90 (1943). [MF 16091]

In geometria affine ad ogni tangente d'una superficie corrisponde un vettore invariantivo dipendente da derivate seconde. Se per tangenti parallele questi vettori sono uguali, la superficie è una quadrica. *Author's summary.*

Dwinger, Ph. Über Normalenkongruenzen. Nieuw Arch. Wiskunde (2) **21**, 81–88 (1941). [MF 15715]

By means of Study's dual numbers the following theorem is proved. If the principal surfaces (Hauptflächen) of a normal congruence intersect the focal surfaces in conjugate directions, then the median (Mittelfläche) is a minimal surface and orthogonal to the lines of the congruence. Necessary and sufficient conditions are given for the focal surfaces to have equal Gauss curvatures at corresponding points. *H. Busemann* (Northampton, Mass.).

Vest, M. L. Involutorial space transformations associated with a linear congruence. Duke Math. J. **12**, 621–628 (1945). [MF 15505]

In a previous paper [Bull. Amer. Math. Soc. **48**, 874–882 (1942); these Rev. **4**, 170] the author studied noninvolutorial transformations associated with a linear congruence. The present paper is concerned with involutorial transformations. The author considers directrices r, s of a linear congruence and a pencil of surfaces $|F|$ such that through a

generic point $P(y)$ there passes a single F of $|F|$. The unique transversal t through $P(y)$, r, and s meets F a second time in a point $Q(x)$. The transformation $P(y)\rightarrow Q(x)$ defines the involution to be studied. The author's main result is that r, s and g (the residual base curve of $|F|$) are fundamental curves of the involution. *N. Coburn.*

Vincensini, Paul. Sur certaines congruences rectilignes appartenant à un complexe linéaire. Ann. Fac. Sci. Univ. Toulouse (4) **4**, 97–115 (1940). [MF 15168]

This is an early edition of the paper reviewed below. Its results are similar but are proved under less general conditions. *C. B. Allendoerfer* (Haverford, Pa.).

Vincensini, Paul. Sur certains types de congruences appartenant à un complexe linéaire, et sur les suites de Laplace de réseaux quadratiques de Wilczynski de période 4. Bull. Soc. Math. France **73**, 1–26 (1945). [MF 14172]

In an earlier paper [Bull. Sci. Math. (2) **65**, 155–178, 186–210 (1941); these Rev. **7**, 31] the author has defined a transformation $T[\mathfrak{F}(Z)]$ which carries a given congruence D into a new congruence Δ. The present paper characterizes completely those normal congruences whose transforms Δ are located on a given linear complex. A further characterization is given of those normal congruences whose transforms lie on a complex and which have either (1) a plane mean surface which is perpendicular to the axis of the complex, or (2) a point mean envelope located on the axis of the complex or (3) associated foci which are equidistant from the axis of the complex. There exists one congruence of the complex common to these three types. Consideration of this congruence leads to a metric study of the periodic sequences of quadratic sets of Wilczynski of period four. All such sequences are determined geometrically. Reference may be made to an earlier draft of this paper [reviewed above] for an elaboration of certain details.

C. B. Allendoerfer (Haverford, Pa.).

Bol, G. Über einen neuen Aufbau der projektiven Flächentheorie. Comment. Math. Helv. **18**, 129–153 (1946).

A generalization of covariant differentiation with respect to a covariant tensor is presented for the study of surfaces in a projective space of three dimensions. If the asymptotic curves are chosen as parametric, the homogeneous coordinates x of a generic point of S satisfy differential equations of the form $x_{uu}=\alpha x_u+\beta x_v+px$, $x_{vv}=\gamma x_u+\delta x_v+qx$. Under the transformation $u=f(\bar{u})$, $v=g(\bar{v})$, β and γ transform according to the law $\bar{\beta}=f'^2\beta/g'$, $\bar{\gamma}=g'^2\gamma/f'$, that is, these invariants have the weights $(2,-1)$, $(-1,2)$, respectively. If, moreover, the coordinates x are so normalized that $(x, x_u, x_v, x_{uv})=1$, then x has the weight $(-\frac{1}{4},-\frac{1}{4})$. Let invariants λ, μ, m of respective weights $(-1,c)$, $(d,-1)$, (a,b) be assumed; then the functions

$$m_1=m_u+am\,(\log\lambda)_u,\quad m_2=m_v+bm\,(\log\mu)_v$$

are invariants with the respective weights $(a+1,b)$, $(a,b+1)$. These functions are called the covariant derivatives of m with respect to λ, μ. In particular, the second covariant derivatives of x are of the form $x_{11}=\beta x_2+lx$, $x_{12}=y+px$, $x_{21}=y+qx$, $x_{22}=\gamma x_1+mx$. The line (xy) so defined is called the first normal of S relative to λ, μ. Similarly, using plane coordinates, a second line lying in the tangent planes is found, called the second normal of S relative to λ, μ. These two lines are reciprocal lines. The congruences generated by these lines are respectively conjugate and harmonic to S if and only if $x_{12}=x_{21}$. If, in particular, $\lambda=\beta^{p-1}\gamma^{2p-1}$, $\mu=\beta^{2p-1}\gamma^{p-1}$, the first normals relative to λ, μ are canonical lines of the first kind. For $p=\frac{2}{3}$, 1, 0, $\frac{1}{2}$ they are, respectively, the first axis of Čech, the first axis of Wilczynski, the Fubini-Green projective normal and the first edge of Green. *V. G. Grove* (East Lansing, Mich.).

Haantjes, J., and Smits, C. The differential geometry of Möbius in the plane. Nieuw Arch. Wiskunde (2) **22**, 34–47 (1943). (Dutch. English summary) [MF 15697]

In this paper the differential geometry of plane curves is developed with respect to the group of conformal transformations, which transform circles into circles (the so-called Möbius-group), by studying the group of linear fractional transformations in a complex variable $z=x+iy$.

From the author's summary.

DeCicco, John. Equilong maps of the ∞^3 circles. Trans. Amer. Math. Soc. **59**, 42–53 (1946). [MF 15317]

Upon applying any equilong transformation to the totality of ∞^3 circles of the plane there results a set of ∞^3 curves which is termed an ω-family. A set of three geometric properties is shown to characterize any ω-family. The first of these properties is typical of the type of result which is obtained: the locus of the foci of the osculating parabolas of the ∞^1 curves of the ω-family which pass through a lineal element E is a cissoid with the cusp at E. An equivalent set of three characteristic properties is also derived. Finally, it is proved that the magnilong group (a magnilong transformation is the product of an equilong transformation by a magnification) consists of all line transformations converting every ω-family into an ω-family. *A. Fialkow.*

Stohler, Gerhard. Über eine Klasse von einparametrigen Differential-Transformationsgruppen. Comment. Math. Helv. **18**, 76–121 (1946).

A complete discussion of "R-transformations" in three variables based on the work of A. Ostrowski [same Comment. **13**, 156–194 (1941); Acta Math. **75**, 151–182 (1943); these Rev. **3**, 43; **7**, 76]. Where Ostrowski deals with a single reversible transformation the present author considers a one-parameter continuous group $\xi=g(x, y_1, y_2, p_1, p_2, a)$, $\eta_\nu=h_\nu(x, y_1, y_2, p_1, p_2, a)$, $\nu=1, 2$, where a is the group parameter. It is shown that the functions determining an R-group are expressible in terms of x, y_1, y_2, a, $r=r(x, y_1, y_2, p_1, p_2)$. Then the four functions of x, y_1, y_2, r form a Lie group which is called the L-group. The paper considers the differential equations that the function r must satisfy and also the conditions on the L-group in order that it should define a group of reversible transformations. Another section of the paper deals with the integral representation of groups of R-transformations and the last section deals with the connection between R-groups and contact transformations, the main theorem being that any B-group, $B_1^{\sigma r}$ contact transformations, leaving $\partial y_1/\partial x=f(\partial y_1/\partial y_2, x, y_1, y_2)$ invariant, determines an endless number of R-groups.

M. S. Knebelman (Pullman, Wash.).

Ehresmann, Charles, et Reeb, Georges. Sur les champs d'éléments de contact de dimension p complètement intégrables dans une variété continuement différentiable V_n. C. R. Acad. Sci. Paris **218**, 955–957 (1944). [MF 15243]

The authors consider fields of p-dimensional elements of contact in a differentiable manifold V_n. An elementary integral manifold is a p-dimensional manifold V_p in V_n such that at every point its tangent space coincides with the element of the field. A field is completely integrable if every

point of V_n lies on some elementary integral. Complete integrals are defined as components of V_n in the topology determined by taking the elementary integrals as neighborhoods. Theorem 1: if V_n is n-space or the n-sphere, and p and $n-1$ are even, then the characteristic of every compact integral manifold of a completely integrable field is 0. The proof uses the "tube" around the manifold, which is a sphere bundle, and the theorem of Hopf about the curvatura integra of a hypersurface. Theorem 2: if K is a completely integrable field of dimension $n-1$, definable by a Pfaffian form and if C_{n-1} is a compact integral with finite fundamental group, then all integrals near C_{n-1} are homeomorphic with it. If also V_n is compact, then all integrals are homeomorphic, and they form a fiber decomposition of V_n; this last statement also holds if all integral manifolds are compact but V_n is not. *H. Samelson* (Syracuse, N. Y.).

Reeb, Georges. Sur les variétés intégrales des champs d'éléments de contact complètement intégrables. C. R. Acad. Sci. Paris **220**, 236–237 (1945). [MF 15232]

The author extends theorem 1 of the paper reviewed above. If V_n is a differentiable manifold and E_p is a p-dimensional completely integrable field of contact elements, then every compact integral manifold which is homologous to 0 has characteristic 0. The proof is similar to the previous one; it involves the assumption that a certain fiber bundle can be so subdivided that the fiber mapping is simplicial. If "homologous" is replaced by "homotopic," this assumption can be dropped. Furthermore, the only manifolds on which there exist differentiable functions with only maxima or minima as singular points are the spheres; there is only one maximum and one minimum, and all level surfaces are spheres. Some more theorems about the existence or nonexistence of compact integrals, in particular, for exact Pfaffian forms, are given. *H. Samelson.*

Wong, Yung-Chow. A note on the first normal space of a V_m in an R_n. Bull. Amer. Math. Soc. **51**, 997–1000 (1945). [MF 14476]

A Riemannian V_m is imbedded in a Euclidean R_n. Through a point P of V_m passes a hypersphere S with its center in the local normal $(n-m)$-plane N. Then S' intersects the V_m in a V_{m-1} with a singular point at P, where the tangent lines to V_{m-1} form a hypercone C in the tangent m-plane. This cone is of the second degree, unless the point P is semi-umbilical, that is, unless a vector v^k exists for which $v_k H^k_{ij} = {}'g_{ij}$, H^k_{ij} being the curvature tensor of V_m and ${}'g_{ij}$ its first fundamental tensor.

If N' is the first normal m'-plane of V_m at P, then all the points in N with the same projection in N' have the same hypercone C. The locus of the points in N' whose hypercone C has a line of vertices is the locus of the intersection of N' at P by the normal $(n-m)$-planes of all the points in the neighborhood of P. This last property is an extension of a theorem of R. Calapso [Rend. Sem. Mat. Univ. Roma (4) **2**, 276–311 (1938)]. *D. J. Struik* (Cambridge, Mass.).

Chern, Shiing-shen. On the curvatura integra in a Riemannian manifold. Ann. of Math. (2) **46**, 674–684 (1945). [MF 14127]

In einer früheren Arbeit des Verfassers [Ann. of Math. (2) **45**, 747–752 (1944); diese Rev. **6**, 106] war für gerades n in jeder n-dimensionalen orientierbaren Riemannschen Mannigfaltigkeit R^n eine (äussere) Differentialform Ω untersucht worden, die durch die Metrik von R^n bestimmt und vom $n$ten Grade, also gleich einem Krümmungsskalar, multipliziert mit dem Volumenelement, ist; es war gezeigt worden, dass Ω als Differentialform in der Mannigfaltigkeit M^{2n-1} der tangentialen Einheitsvektoren von R^n aufgefasst werden kann und dass sie dort die äussere Ableitung einer Form Π des Grades $n-1$ ist. Jetzt wird Π gleichzeitig für alle n, gerade und ungerade, definiert, für ungerades n wird $\Omega=0$ gesetzt, und es gilt $-d\Pi=\Omega$ (bei geradem n ist das Vorzeichen von Ω jetzt anders gewählt als früher).

Es wird nun erstens für ein Teilpolyeder P^n von R^n, dessen Rand ∂P^n differenzierbar ist, ein neuer Beweis der Formel von Allendoerfer-Weil [Trans. Amer. Math. Soc. **53**, 101–129 (1943); diese Rev. **4**, 169]

$$(1) \qquad \int_{P^n} \Omega + \int_{\partial P^n} \Pi = \chi'(P^n) = (-1)^n \chi(P^n)$$

geliefert; hierin sind die Integrale über ein Feld von Einheitsvektoren zu erstrecken, das auf ∂P^n normal nach innen gerichtet ist und in P^n höchstens endlich viele Singularitäten besitzt; χ bezeichnet die Eulersche Charakteristik, χ' die innere Charakteristik (also $\chi(P)=\chi'(P)+\chi(\partial P)$). Die Formel (1) enthält die Formel

$$(1') \qquad \int_{R^n} \Omega = \chi(R^n),$$

deren Beweis das Ziel der zitierten früheren Arbeit des Verfassers war.

Zweitens wird (dies ist das Hauptziel der Arbeit) für jede orientierbare geschlossene Teilmannigfaltigkeit R^m von R^n, $m<n$, die Formel

$$(2) \qquad -\int_{R^m} \Pi = \chi(R^m)$$

bewiesen, in welcher das Integral über die Mannigfaltigkeit aller Normalvektoren (der Länge 1) von R^m zu erstrecken ist (beim Vergleich der Formel (2) mit der Formel (1) für $R^m=\partial P^n$ beachte man, dass in (1) nur über das innere, in (2) über das innere und über das äussere Normalenfeld zu integrieren ist). Beim Beweis von (2) wird (1), angewandt auf R^m statt R^n, benutzt. Der Verfasser bemerkt, dass (2) selbst für den Fall, in dem R^n der euklidische Raum ist, neu zu sein scheint.

Drittens wird eine geschlossene orientierbare R^n in einer orientierbaren R^{2n} betrachtet. Nach Whitney [Lectures in Topology, pp. 101–141, University of Michigan Press, 1941; diese Rev. **3**, 133] ist die Indexsumme ψ der Singularitäten eines Normalenfeldes von R^n eine Invariante der Lage von R^n in R^{2n}. Der Verfasser definiert, in Analogie zu der Definition der Krümmungsinvariante Ω, eine Torsionsinvariante Θ von R^n, die bei ungeradem n gleich 0 ist; analog zu dem Beweise von (1') lässt sich die Formel

$$(3) \qquad \int_{R^n} \Theta = \psi$$

beweisen. Aus ihr folgt leicht: Ist n ungerade, so existiert auf R^n ein Normalenfeld ohne Singularität. *H. Hopf.*

Abe, Makoto. Sur la réductibilité du groupe d'holonomie. I. Les espaces à connexion affine. Proc. Imp. Acad. Tokyo **20**, 56–60 (1944). [MF 14871]

In an affinely connected n-space, Cartan's holonomic group associated with a point O may be reducible and leave invariant one or more multi-directions (multi-planes in the tangent space at O). To each invariant p-direction there corresponds in the n-space an absolutely parallel field of

p-directions, and conversely. In space without torsion the p-planes of such a field are the tangent planes of an $(n-p)$-parameter family of totally geodesic subspaces.

A. G. Walker (Liverpool).

Abe, Makoto. Sur la réductibilité du groupe d'holonomie. II. Les espaces de Riemann. Proc. Imp. Acad. Tokyo **20**, 177–182 (1944). [MF 14880]

[Cf. the preceding review.] If the holonomic group associated with a point of a Riemannian space V^n leaves a p-direction invariant, then V^n is reducible and is the product of two Riemannian spaces V^p, V^{n-p}, that is, is such that its metric can take the form

$$ds^2=g_{ij}du^idu^j+g_{\alpha\beta}du^\alpha du^\beta,$$
$$i,j=1,\cdots,p;\ \alpha,\beta=p+1,\cdots,n,$$

where g_{ij}, $g_{\alpha\beta}$ do not contain u^α, u^i, respectively. The manner in which a V^n can be completely decomposed is unique. A V^n is the product of a Euclidean R^p and a Riemannian V^{n-p} if and only if it admits p linearly independent absolutely parallel vector fields. *A. G. Walker.*

Kodaira, Kunihiko. Über die Harmonischen Tensorfelder in Riemannschen Mannigfaltigkeiten. I. Proc. Imp. Acad. Tokyo **20**, 186–198 (1944). [MF 14882]

Let M be a Riemannian manifold of n dimensions, with metrical tensor g_{ij}. If $P^p=P_{i_1\cdots i_p}$ is a skew-symmetric covariant tensor field, tensor fields $\mathfrak{d}P$, τ^*P, τP of ranks $n-p$, $p+1$, $p-1$, respectively, are defined as follows:

$$\mathfrak{d}P=(1/p!)\sqrt{(g)}\epsilon_{i_1\cdots i_{n-p}j_1\cdots j_p}g^{j_1h_1}\cdots g^{j_ph_p}P_{h_1\cdots h_p},$$
$$\tau^*P=\partial_{[i_1}P_{i_2\cdots i_{p+1}]},$$
$$\tau P=(-1)^{np}\mathfrak{d}\tau^*\mathfrak{d}P.$$

The field P^p is harmonic if $\tau^*P=0$, $\tau P=0$. This paper gives a new proof of the existence of a harmonic tensor field of rank p which is regular at all points of M and is such that $\int P_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}$ has assigned periods on the cycles of a base for the p-cycles on M [cf. W. V. D. Hodge, The Theory and Applications of Harmonic Integrals, Cambridge University Press, 1941; H. Weyl, Ann. of Math. (2) **44**, 1–6 (1943); these Rev. **2**, 296; **5**, 37]. The method is that of orthogonal projection in Hilbert space.

The relation between the conception of harmonic tensors and the topology of the Riemannian manifold is examined [cf. Hodge, op. cit., pp. 88–92]. The method of orthogonal projection as given by Weyl [Duke Math. J. **7**, 411–444 (1940); these Rev. **2**, 202] is then applied to prove the necessary existence theorem. As in Weyl's work, the difficulty lies in proving the analyticity of the solution; this is achieved by a generalisation of the method used by Weyl for harmonic functions in Euclidean 3-space.

W. V. D. Hodge (Cambridge, England).

Kodaira, Kunihiko. Über die Harmonischen Tensorfelder in Riemannschen Mannigfaltigkeiten. II. Proc. Imp. Acad. Tokyo **20**, 257–261 (1944). [MF 14890]

The result of the paper reviewed above is extended to harmonic tensors with singularities. Let F be any closed subset of the Riemannian manifold M and let U be a neighbourhood of F. If Φ is a tensor field of rank p, defined in U, which is harmonic in $U-F$, it is proved that there exists a tensor field e defined on M and regular on $M-F$ such that $e-\Phi$ is harmonic and regular everywhere in U. In the cases $p=1$, $n-1$ a condition on Φ must be imposed similar to the condition that the sum of the polar periods of an Abelian integral on a Riemann surface is zero but in other cases no restriction is imposed. An application is made to the theory of integrals on a Riemann surface.

W. V. D. Hodge (Cambridge, England).

Kodaira, Kunihiko. Über die harmonischen Tensorfelder in Riemannschen Mannigfaltigkeiten. III. Proc. Imp. Acad. Tokyo **20**, 353–358 (1944). [MF 14901]

[Cf. the preceding review.] It is proved that there exist, on a Riemannian manifold M, two sets h_1^p, h_2^p, $\cdots$ and h_1^{*p}, h_2^{*p}, $\cdots$ of tensor fields of rank p (for $p=0,\cdots,n$) such that

$$\tau^*h_j^p=\chi_j^ph_j^{*p+1},\ \tau h_j^p=0,$$
$$\tau h_j^{*p+1}=\chi_j^ph_j^p,\ \tau^*h_j^{*p+1}=0,$$

where χ_1^p, χ_2^p, $\cdots$ are real numbers such that

$$0<\chi_1^p\leqq\chi_2^p\leqq\cdots,\quad\lim_{j\to\infty}\chi_j^p=+\infty.$$

If $e_1^p,\cdots,e_{p_p}^p$ is a suitably chosen basis for the harmonic tensor fields on M, then $e_1^p,\cdots,e_{p_p}^p$, $h_1^p,\cdots,h_1^{*p},\cdots$ is a complete orthogonal basis for the tensor fields $\phi_{i_1\cdots i_p}$ on M such that $\sqrt{(g)}g^{i_1j_1}\cdots g^{i_pj_p}\phi_{i_1\cdots i_p}\phi_{j_1\cdots j_p}dx^1\cdots dx^n$ is Lebesgue integrable on M. *W. V. D. Hodge.*

Slebodziński, W. Sur une dégénérescence de la variété riemannienne. Revista Ci., Lima **47**, 423–433 (1945). [MF 15429]

The problem is to find an affine connection intrinsically associated with a two-dimensional Riemann space whose metric is degenerate; specifically, $g_{11}=g_{12}=0$, $g_{22}=1$. If one insists that the resultant affinely connected space (a) possess an absolute unit of area and (b) be symmetric in the sense of É. Cartan, there are three solutions, one of them affinely flat, the other two projectively flat and admitting a 3-parameter group of automorphisms. A canonical form is given for the components of connection.

J. L. Vanderslice (College Park, Md.).

Potier, R. Sur certaines questions de géométrie différentielle conforme. Ann. Fac. Sci. Univ. Toulouse (4) **4**, 1–63 (1940). [MF 15166]

The author's principal object is the study of curves on a surface in a conformally connected 3-space. He takes this opportunity to make effective use of Cartan's methods to develop in considerable detail those aspects of conformal theory bearing on his problem. The first three chapters are restricted to classical conformal 3-space. An intrinsic pentaspherical reference system is associated with each point of a curve, resulting in Frenet equations and invariants of curvature and torsion. When the curve is replaced by a surface an analogous procedure leads also to intrinsic local reference frames with a resulting conformal displacement along the surface. These two studies are combined into a treatment of curves on surfaces; relationships between the reference systems and their invariants are exhibited and in many cases interpreted geometrically. Conformal asymptotic lines and geodesics are defined and some interesting properties derived. Several theorems on isometric conformal deformation of isometric surfaces also emerge. To quote the simplest: the torsions of the lines of curvature are invariant. In chapter IV the notion of conformally connected space is introduced and illustrated by showing how a connection is induced on a surface in ordinary conformal space. Using this, the intrinsic geometry of curves on the surface is developed and the results compared with those of earlier chapters. Here the notion of generalized circle replaces that

of conformal geodesic. Finally, chapter V treats the case of a surface in conformally connected 3-space. The torsion of the space introduces notable differences from the previous theory. For example, the lines of curvature on a surface are in general no longer orthogonal and the relations found earlier between the reference systems and invariants are modified. *J. L. Vanderslice* (College Park, Md.).

Sasaki, Shigeo. On the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere. I. Jap. J. Math. **18**, 615–622 (1943). [MF 14975]

The author proves that, if the group of holonomy of a C_n, a space with a normal conformal connection, fixes a point (or a hypersphere), then the C_n corresponds to a set of Riemann spaces conformal to each other and including an Einstein space with vanishing (or nonvanishing) scalar curvature, and conversely. Using this basic theorem to refer questions about Riemann spaces conformal to Einstein spaces to the corresponding holonomy groups, the author derives several theorems derived earlier by H. W. Brinkmann [Math. Ann. **91**, 269–278 (1924)]. For instance, if a Riemann space V_n is conformal to an Einstein space in $r>1$ ways, the V_n is conformal to an Einstein space with vanishing scalar curvature. *A. Schwartz* (State College, Pa.).

Sasaki, Shigeo. On the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere. II. Jap. J. Math. **18**, 623–633 (1943). [MF 14976]

The basic theorem of the paper reviewed above enables the author to study the Einstein spaces conformal to other Einstein spaces by referring to C_n's whose holonomy groups fix two hyperspheres. It is shown that, if the holonomy group fixes two hyperspheres, the line element of the C_n can be reduced to the canonical form

$$ds^2=[(x^n)^2+K]^2 g^*_{bc}(x^a)dx^b dx^c+(dx^n)^2,$$

where K is a constant greater than, equal to or less than 0 according as the pencil of hyperspheres determined by the invariant hyperspheres is hyperbolic, parabolic or elliptic, and the Riemann space with the fundamental tensor $g^*_{ab}(x^a)$ is an Einstein space. The converse is also true. Another of the theorems proved states a sufficient condition that a C_n whose holonomy group fixes a real hypersphere shall admit an umbilical hypersurface. Such C_n's are called C-complete. Many of the results were obtained by Brinkmann [Math. Ann. **94**, 119–145 (1925)], but the reasoning used here is of a more geometric nature. *A. Schwartz.*

Sasaki, Shigeo. On the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere. III. Jap. J. Math. **18**, 791–795 (1943). [MF 14985]

The author pointed out in the papers reviewed above that, if there is an Einstein space with negative scalar curvature in the set of Riemann spaces conformal to each other which corresponds to a C-complete C_n, then this C_n admits an umbilical hypersurface U. Here it is shown that the distance between two points P_0 and P_s on a geodesic C of the Einstein space (C is a conformal circle of the C_n) is equal to $\{n(n-1)/(-R)\}^{\frac{1}{2}}$ times the natural logarithm of the cross-ratio of the four values of the projective parameter of C corresponding to P_0, P_s and the points at which C cuts the umbilical hypersurface U. Thus there exists a representation for Einstein spaces with negative scalar curvature analogous to Poincaré's representation for non-Euclidean geometries. *A. Schwartz* (State College, Pa.).

Yano, Kentaro. Conformal and concircular geometries in Einstein spaces. Proc. Imp. Acad. Tokyo **19**, 444–453 (1943). [MF 14841]

Sasaki [see the three preceding reviews] has studied spaces C_n with normal conformal connections whose groups of holonomy either (A) fix a point or a hypersphere or (B) fix two points or hyperspheres. The author considers an exceptional situation under case (A). He also studies the relations between the conformal and concircular geometries for spaces arising under case (B). The following theorems are typical. (1) The C_n whose group of holonomy fixes two points or hyperspheres is conformal to an Einstein space which admits concircular transformations. (2) The Riemannian spaces which contain a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are conformal circles are conformal to the Riemannian spaces which admit concircular transformations. *A. Fialkow.*

Yano, Kentaro, et Sasaki, Shigeo. Sur les espaces à connexion conforme normale dont les groupes d'holonomie fixent une sphère à un nombre quelconque de dimensions. I. Proc. Imp. Acad. Tokyo **20**, 525–535 (1944). [MF 14919]

The following theorem is proved. The holonomy group of a space C_n with a conformal normal connection fixes an $(m-1)$-dimensional sphere if and only if the fundamental quadratic form of C_n may be written as

$$ds^2=f(x^\lambda)g_{bc}(x^a)dx^b dx^c+h(x^\lambda)g_{ji}(x^c)dx^j dx^i,$$

where $\lambda=1, 2, \cdots, n$; $a, b, c=1, 2, \cdots, m$; $i, j, k=m+1, m+2, \cdots, n$. Various geometric properties of these spaces are derived. *A. Fialkow* (New York, N. Y.).

Yano, Kentaro. Concircular geometry. V. Einstein spaces. Proc. Imp. Acad. Tokyo **18**, 446–451 (1942). [MF 14771]

[Part IV appeared in the same Proc. **16**, 505–511 (1940); these Rev. **2**, 303.] A concircular transformation of a Riemann space V_n is a transformation which leaves the geodesic circles of V_n invariant. In this paper the author studies the properties of Einstein spaces which admit concircular transformations. Making use of some of his earlier results in this field, he obtains the first fundamental form of any Einstein space which admits a concircular transformation. It is also shown that, if an Einstein space is conformal to an Einstein space, it must admit a concircular transformation. Consequently the present work is closely related to that of Brinkmann [Math. Ann. **94**, 119–145 (1925)] on Einstein spaces which are mapped conformally on each other. *A. Fialkow* (New York, N. Y.).

Yano, Kentaro. Les espaces d'éléments linéaires à connexion projective normale et la géométrie projective générale des paths. Proc. Phys.-Math. Soc. Japan (3) **24**, 9–25 (1942). [MF 15018]

In generalizing the notion of paths J. Douglas has employed equations of the form $d^2u^i/ds^2+2\Gamma^i(u, du/ds)=0$ [equations (5.8) of the present paper], where the Γ^i are homogeneous functions of degree two with respect to the arguments du^i/ds. Following the general methods of É. Cartan [Bull. Soc. Math. France **52**, 205–241 (1924)] the author geometrizes from the projective point of view equations of this type for the case of n dimensions. In the manner of Cartan, spaces of linear elements with projective

connection are defined and the effects of the following transformations on the components of the connection are determined: (i) transformations of the plane at infinity of the local reference frame, (ii) transformations of coordinates u^i. The conditions that the space is without torsion are then obtained by application of the usual definition.

Let ρA_θ denote general homogeneous coordinates of an arbitrary point A_θ of the space. The author defines the paths by equations of the form $d^2(\rho A_\theta)/dt^2=0$. Making use of the differential equations which define the projective connection for the space, the equations for the paths are expressed in terms of the components of the projective connection. At this point the problem is proposed of constructing a space of linear elements with projective connection whose paths are defined by a system of equations (5.8). To solve this problem the tensors of curvature of the space are calculated. Three of these tensors have projective significance. On imposing the intrinsic conditions that certain contractions of two of these tensors form null tensors the connection becomes, in the sense of Cartan, a normal projective connection. Finally, components of a normal projective connection are calculated in terms of the partial derivatives of Γ^i with respect to the arguments du^i/ds.

P. O. Bell (Lawrence, Kan.).

Yano, Kentaro. Projective parameters in projective and conformal geometries. Proc. Imp. Acad. Tokyo **20**, 45–53 (1944). [MF 14869]

In the first part of the paper the author gives a survey of the different methods of introducing a projective parameter on a path in a projective path space, in particular, the methods used by T. Y. Thomas, who introduces an additional coordinate [Proc. Nat. Acad. Sci. U. S. A. **11**, 199–203 (1925)], L. Berwald, who defines a projective parameter by means of a Schwarzian derivative [Ann. of Math. (2) **37**, 879–898 (1936)] and J. Haantjes, who uses homogeneous coordinates [Proc. Edinburgh Math. Soc. (2) **5**, 103–115 (1937)]. In the second part the author shows that the method of Berwald may be applied to conformal geometry. It is shown that the parameter t defined by the Schwarzian derivative

$$\{t, s\}=\tfrac{1}{2}a_{ih}\frac{\delta^2x^i}{ds^2}\frac{\delta^2x^h}{ds^2}+\frac{1}{n-2}L_{ih}\frac{dx^i}{ds}\frac{dx^h}{ds},$$

where the tensor L_{ij} is derived from the curvature affinor as follows,

$$L_{ij}=-K_{ij}+\{2(n-1)\}^{-1}Ka_{ij},\quad K_{ij}=K_{hij}^{\ \ \ h},\quad K=K_{ij}a^{ij},$$

is a conformal invariant. It is defined on each curve up to a linear fractional transformation. *J. Haantjes.*

Yano, Kentaro. Über eine geometrische Deutung der projektiven Transformationen nicht-symmetrischer affiner Übertragungen. Proc. Imp. Acad. Tokyo **20**, 284–287 (1944). [MF 14896]

A necessary and sufficient condition that two nonsymmetric affine connections L^i_{jk} and $\bar{L}^i_{jk}$ be related by the equation

$$\bar{L}^i_{jk}=L^i_{jk}+p_j\delta_k^{\ i}+q_k\delta_j^{\ i}$$

is derived. The characterization is given in terms of a vector $v^i(s)$ defined along a curve $x^i(s)$ such that, if the curve is developed upon the tangent space at some point, the directions $v^i(s)$ form a developable surface or torse. The vectors v^i are termed "torse-forming directions."

A. Fialkow (New York, N. Y.).

Yano, Kentaro. On the torse-forming directions in Riemannian spaces. Proc. Imp. Acad. Tokyo **20**, 340–345 (1944). [MF 14898]

Some properties of torse-forming directions [see the preceding review] in a Riemann space V_n are derived. It is proved that V_n admits a torse-forming vector field if and only if it contains a family of ∞^1 parallel totally umbilical hypersurfaces. The expression for the fundamental quadratic form of this V_n is given. *A. Fialkow.*

Yano, Kentaro. Subprojective transformations, subprojective spaces and subprojective collineations. Proc. Imp. Acad. Tokyo **20**, 701–705 (1944). [MF 14945]

Let $\Pi^\lambda_{\mu\nu}(x)$ be the coefficients of connection of an affinely connected space A_n and $\xi^\lambda(x)$ a contravariant vector field of A_n. The curves which are solutions of

$$\frac{d^2x^\lambda}{dr^2}+\Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=\alpha\frac{dx^\lambda}{dr}+\beta\xi^\lambda$$

are termed "subpaths with respect to ξ^λ." A subprojective change of affine connection is the most general change of $\Pi^\lambda_{\mu\nu}$ which leaves these subpaths invariant. The law for a subprojective change is derived. It is shown that, if $\Pi^\lambda_{\mu\nu}$ and $\overline{\Pi}^\lambda_{\mu\nu}$ are subprojectively equivalent and ξ^λ is torse-forming [see the second preceding review] with respect to one of them, then it is torse-forming with respect to the other. Subprojective spaces and collineations are also defined and their properties studied. *A. Fialkow* (New York, N. Y.).

Yano, Kentaro. Sur les espaces à connexion affine qui peuvent représenter les espaces projetifs des paths. Proc. Imp. Acad. Tokyo **20**, 631–639 (1944). [MF 14935]

Two sets of tensor conditions are derived which characterize an $(n+1)$-dimensional space A_{n+1} with an affine connection which can represent an n-dimensional projective space of paths. In particular, A_{n+1} must contain a torse-forming vector field [see the third preceding review].

A. Fialkow (New York, N. Y.).

Yano, Kentaro. Sur les espaces à connexion affine qui peuvent représenter les espaces projectifs des paths. II. Proc. Imp. Acad. Tokyo **21**, 16–24 (1945). [MF 14950]

The results of the paper reviewed above are derived by new proofs which do not depend upon any special coordinate systems. Some properties of the projective parameters on the paths of the space are given. *A. Fialkow.*

Yano, Kentaro. Sur les espaces à connexion affine qui peuvent représenter les espaces projectifs des paths. III. Proc. Imp. Acad. Tokyo **21**, 97–103 (1945). [MF 14954]

[Cf. the preceding review.] The author studies certain n-dimensional projective spaces of paths P_n which may be represented by $(n+1)$-dimensional spaces with an affine connection. He finds a necessary and sufficient condition that the paths of P_n admit projective parameters.

A. Fialkow (New York, N. Y.).

Yano, Kentaro, et Mutô, Yosio. Sur la théorie des hypersurfaces dans un espace à connexion conforme. Jap. J. Math. **17**, 229–288 (1941). [MF 14958]

In an earlier paper [J. Fac. Sci. Imp. Univ. Tokyo Sect. I. **4**, 1–59 (1939); these Rev. **1**, 88], Yano elaborated part of E. Cartan's work on spaces with conformal connection [Ann. Soc. Polon. Math. **2**, 171–221 (1924)]. This paper continues the elaboration of Cartan's work. Yano's earlier paper is reviewed; the authors then turn to a hypersurface

in a space with conformal connection defined through parametric equations. Polyspherical coordinates are set up at each point of the hypersurface and a conformal connection for the hypersurface is determined by that of the parent space. A conformal curvature tensor for the hypersurface is studied and equations analogous to the Gauss and Codazzi equations are developed. Curves in the hypersurface can also be considered as curves in the parent space, and the relationships between the two projective parameters and the two sets of conformal curvatures are studied. For umbilical surfaces the two projective parameters coincide except perhaps for a homographic transformation. The differential equations of the generalized circles on the hypersurface (curves of vanishing first conformal curvature) and those of some special curves defined by A. Haimovici [C. R. Acad. Sci. Roum. 1, 296–301 (1937)] are derived. Finally, there are some special theorems dealing with umbilical points and umbilical hypersurfaces. *A. Schwartz.*

Yano, Kentaro, and Mutô, Yosio. On the conformal arc length. Proc. Imp. Acad. Tokyo 17, 318–322 (1941). [MF 14713]

The Frenet equations for a curve in a conformally connected manifold as previously derived by K. Yano [J. Fac. Sci. Imp. Tokyo Univ. Sect. I. 4, 1–59 (1939); these Rev. 1, 88] were not completely invariant. Starting with them the authors find a conformal invariant of least degree which, used as a definition of conformal arc length, leads to invariant Frenet formulae involving certain conformal scalar invariants $\lambda, \lambda^{(1)}, \cdots, \lambda^{(n+1)}$ as curvatures.

J. L. Vanderslice (College Park, Md.).

Yano, Kentaro, and Mutô, Yosio. On the generalized loxodromes in the conformally connected manifold. Proc. Imp. Acad. Tokyo 17, 455–460 (1941). [MF 14726]

This is a sequel to the paper reviewed above. Any curve for which $\lambda=$ constant, the other λ's being 0, when developed on one of the conformal tangent spaces lies on a two-dimensional sphere and cuts the circles through two fixed points at an angle $\pi/4-\frac{1}{2}\tan^{-1}\lambda$. Thus these curves are generalized loxodromes. In the case $\lambda=0$ their differential equations (fourth order) are derived. *J. L. Vanderslice.*

Yano, Kentaro, and Mutô, Yosio. On the curves developable on two-dimensional spheres in the conformally connected manifold. Proc. Imp. Acad. Tokyo 18, 222–226 (1942). [MF 14756]

This is a sequel to the paper reviewed above. The case now considered is that of any curve C for which $\lambda=f(\sigma)$, other λ's $=0$, where σ is conformal arc length. The principal result is a geometrical interpretation of λ provided by the following theorem in which the curve C has been developed on the tangent space at one of its points and thus lies on a two-dimensional sphere: if a circle and its four consecutive circles in a coaxial system do not cut each other and cut C at an angle $\alpha<\pi/2$ then at the point $\tan 2\alpha=-1/\lambda$.

J. L. Vanderslice (College Park, Md.).

Yano, Kentaro, et Mutô, Yosio. Sur le théorème fondamental dans la géométrie conforme des sous-espaces riemanniens. Proc. Phys.-Math. Soc. Japan (3) 24, 437–449 (1942). [MF 15031]

The authors try to find a conformal differential geometry analogue of the fundamental theorem of the surface theory of Riemannian geometry as stated by L. P. Eisenhart [Riemannian Geometry, Princeton University Press, 1926, p. 192]. Paralleling the method used by Eisenhart and by others in Riemannian geometry, the authors find five equations which are necessary and sufficient conditions that a symmetric tensor $g_{\mu\lambda}$, $n-m$ symmetric tensors $M_{\mu\lambda P}$, and $(n-m)(n-m-1)/2$ vectors $L_{PQ\lambda}$ $(=-L_{QP\lambda})$ shall determine a subspace of m dimensions, imbedded in a Euclidean space of n dimensions, whose fundamental conformal tensors are $\rho^2 g_{\mu\lambda}$, $\rho M_{\mu\lambda P}$ and $L_{PQ\lambda}$. Three of these equations are analogues of the Gauss, Codazzi and Ricci equations of Riemannian geometry. *A. Schwartz* (State College, Pa.).

Yano, Kentaro. Sur les équations fondamentales dans la géométrie conforme des sous-espaces. Proc. Imp. Acad. Tokyo 19, 326–334 (1943). [MF 14827]

A subspace of an n-dimensional conformally connected manifold has two conformal connections associated with it: one induced by the conformal displacement in the ambient space by virtue of the chosen "rigging" of the subspace, the other intrinsically determined by the induced fundamental conformal tensor. These connections are identical when and only when a certain tensor C_{ij} vanishes. Part of the paper is devoted to finding the explicit form of C_{ij} and the associated scalar C in terms of the first and second fundamental tensors and the curvature tensor. The remainder of the paper derives the fundamental differential equations of the subspace together with equations analogous to those of Gauss, Codazzi and Ricci. *J. L. Vanderslice.*

Yano, Kentaro. Sur une application du tenseur conforme $C_{\mu\lambda}$ et de la scalaire conforme C. Proc. Imp. Acad. Tokyo 19, 335–340 (1943). [MF 14828]

A series of theorems follow from the form of the quantities C_{ij} and C found in the paper reviewed above. For example, a necessary and sufficient condition that the induced and intrinsic connections of a hypersurface in conformally flat space coincide is that the hypersurface be totally umbilical and hence conformally flat. Incidentally, the scalar C is shown to be identical with the scalar λ of S. Sasaki [Sci. Rep. Tôhoku Imp. Univ., Ser. 1. 29, 219–267 (1940); these Rev. 2, 165]. *J. L. Vanderslice.*

Yano, Kentaro, and Adati, Tyûzi. Parallel tangent deformation, concircular transformation and concurrent vector field. Proc. Imp. Acad. Tokyo 20, 123–127 (1944). [MF 14878]

A "parallel tangent deformation" is an infinitesimal transformation which displaces the tangent to a curve in a parallel direction (in Levi-Civita's sense) at each point. A "concircular transformation" of a Riemann space is one which leaves invariant the geodesic circles of the space. It is proved that a necessary and sufficient condition that a Riemann space V_n $(n>2)$ admit an infinitesimal deformation which gives a parallel tangent deformation for any curve in V_n is that V_n admit a concircular transformation. A condition that a V_n admit a concurrent vector field is also obtained. *A. Fialkow* (New York, N. Y.).

Yano, Kentaro. Projective parameters on paths in D. van Dantzig's projective space. Proc. Imp. Acad. Tokyo 20, 210–215 (1944). [MF 14885]

The author discusses the projective geometry of paths with the use of homogeneous curvilinear coordinates [cf. J. Haantjes, Proc. Edinburgh Math. Soc. 5, 103–115 (1937)]. The paths are given by the differential equations

$$\frac{d^2x^\kappa}{dp^2}+\Pi^\kappa_{\lambda\mu}\frac{dx^\mu}{dp}\frac{dx^\lambda}{dp}=\alpha\frac{dx^\kappa}{dp}+\beta x^\kappa.$$

It is possible to choose a function ρ and a parameter t for which these equations take the form

$$\frac{d^2\rho x^\alpha}{dt^2}+\Pi^\alpha_{\lambda\mu}(\rho x)\frac{d\rho x^\mu}{dt}\frac{d\rho x^\lambda}{dt}=0.$$

It is shown that this parameter t coincides with the parameter used by Berwald and Haantjes [L. Berwald, Ann. of Math. (2) **37**, 879–898 (1936)]. *J. Haantjes.*

Kanitani, Jôyô. Sur les espaces représentables géodésiquement sur l'espace projectif. Proc. Phys.-Math. Soc. Japan (3) **25**, 87–100 (1943). [MF 15050]

The author studies spaces with nonnormal projective connection which admit totally geodesic surfaces through each point tangent to arbitrary plane elements at the point. Let S be a totally geodesic surface, immersed in a projectively connected space $\mathfrak{R}$ and representable geodesically on a projective space. Let C denote a nongeodesic curve on S passing through a point M of S, let Γ denote the development of C in the projective space $\mathfrak{P}$ tangent to $\mathfrak{R}$ at M and let Γ' denote the development of C by means of the projective normal connection to which the nonnormal connection is transformed with preservation of geodesics. We may now summarize the principal results of the paper as follows.

(1) If $\mathfrak{R}$ is a space with torsion, the nongeodesic curves on S cannot be developed on plane curves in the projective tangent space $\mathfrak{P}$ at M. However, the osculating plane at M of the development Γ of such a curve coincides with the tangent plane Π of S at M. A necessary and sufficient condition that the tangent plane Π have third order contact with Γ is that the development of every infinitesimal closed circuit starting at M and returning to M be a curve which terminates in a point on the tangent plane Π.

(2) If the development, by means of the nonnormal projective connection, on $\mathfrak{P}$ of every infinitesimal closed circuit through M, whose initial point and terminal point coincide in M, is a curve which terminates in a point of the tangent plane Π, and if every such circuit on S is developed on a curve which terminates in a point of the tangent to C at M, the curve Γ has contact of third order with the curve Γ' at the point M.

(3) A space with projective normal connection is equivalent to a projective space if and only if through each point M of the former space a quadric Q may be passed in such a manner that the developments in the tangent projective space at M of the curves on Q have contact of third order with an arbitrarily given quadric which passes through M and lies in the projective space. [The author gives credit to Kimpara for having presented this theorem a number of years earlier.]

(4) Let S be a quadric taking the role of S in (1) above. Consider a nonasymptotic curve C of S which has as tangent line at M an asymptotic tangent T of S. The development Γ' of every curve of S lies on a quadric Q in $\mathfrak{P}$. The curve Γ has third order contact with Q at M if and only if it has third order contact with Π at M. *P. O. Bell.*

Kanitani, Jôyô. Sur un espace à connexion projective renfermant des hyperquadriques. Proc. Phys.-Math. Soc. Japan (3) **25**, 617–621 (1943). [MF 15073]

In this note the author proves the following generalization of a theorem appearing in the paper reviewed above. An n-dimensional space with normal projective connection is equivalent to a projective space if and only if one can pass through each point of the first space a hyperquadric V^2_{n-1} in such a manner that the curves on it, developed in the tangent projective space at this point, have contact of third order with an arbitrarily given quadric which passes through the point and lies on the tangent projective space. *P. O. Bell* (Lawrence, Kan.).

Tsuboko, Matsuji. On the projective deformation of curves in E_4. Proc. Phys.-Math. Soc. Japan (3) **25**, 147–168 (1943). [MF 15053]

By extending J. Kanitani's definition of the projective deformation of curves in three-dimensional projective space [Mem. Ryojun Coll. Engin. **6**, 147–154 (1933)] the author obtains the following definition. A point correspondence between the points x, y of two curves $\Gamma(x)$, $\Gamma(y)$ in four-dimensional projective space is a projective deformation if there exists in association with each point on $\Gamma(x)$ a projective transformation (T) which satisfies the following three conditions. (1) A point P on $\Gamma(x)$ is transferred by (T) to the position of the corresponding point P_1 on $\Gamma(y)$. (2) Let Q_1 be the point on $\Gamma(y)$ corresponding to a point Q in the neighborhood of P on $\Gamma(x)$, and Q' the point to which Q is transferred by (T). Then (T) transforms the curve $\Gamma(x)$ to such a position that the distance $[Q_1Q']$ is an infinitesimal of the sixth order with respect to $[P_1Q_1]$ when Q tends to P. (3) The transformation (T) transforms the "osculating quartic curve" of $\Gamma(x)$ into that of $\Gamma(y)$ at the corresponding points.

Let reference frames "of coincidence" [Kanitani, Mem. Ryojun Coll. Engin. **9**, 29–39 (1936)] be associated with x on $\Gamma(x)$ and with y on $\Gamma(y)$, and let x and y correspond to the same values of a parameter u. Then the coefficients of the associated systems of differential equations of $\Gamma(x)$ and $\Gamma(y)$ are (except for constant multipliers) given by corresponding sets of functions h, k, θ_3, θ_4, θ_5 and h_1, k_1, Φ_3, Φ_4, Φ_5, respectively. The following theorem is proved. Necessary and sufficient conditions that the point correspondence between the curves $\Gamma(x)$ and $\Gamma(y)$ be a projective deformation are that the equations $h=h_1$, $k=k_1$, $\theta_3=\Phi_3$, $\theta_4=\Phi_4$ be satisfied. For each case of a number of types of curves defined by the vanishing of certain coefficients of the associated differential equations the number of projective deformations of the curve on itself is determined. A typical result of this character is the following. A curve $\Gamma(x)$ whose tangents intersect an osculating solid of $\Gamma(x)$ along a twisted cubic admits ∞^3 projective deformations on itself. Such a curve is characterized by the conditions $\theta_3=\theta_4=0$.

In the concluding sections the author derives a number of geometric characterizations of the projective deformation between curves of unspecialized type. The following characterization is representative. Let x, y and P, Q be two pairs of corresponding points on $\Gamma(x)$ and $\Gamma(y)$ and P' and Q' the points where the tangents at P and Q of $\Gamma(x)$ and $\Gamma(y)$ intersect the osculating solids at x and y of those curves, respectively. In order that the point correspondence between $\Gamma(x)$ and $\Gamma(y)$ be a projective deformation, it is necessary and sufficient that, by a proper projective transformation which transfers x to y and at the same time the osculating solid at x of $\Gamma(x)$ to that at y of $\Gamma(y)$, P' can be transferred to such a position P_1' that the distance $[P_1'Q']$ is an infinitesimal of the sixth order with respect to $[yQ']$. *P. O. Bell* (Lawrence, Kan.).

Tomonaga, Yasuro. On the theory of hypersurfaces in the path-space of the third order. Proc. Imp. Acad. Tokyo **19**, 341–347 (1943). [MF 14829]

The theory of spaces of paths of third order was developed by Hombu [same Proc. **13**, 187–190 (1937); **14**, 36–40

(1938)]. The present paper deals with various properties of a hypersurface V_{n-1} immersed in this path-space. The analogues of the Gauss-Codazzi equations for V_{n-1} are derived and simple geometric consequences are noted.

A. Fialkow (New York, N. Y.).

Hombu, Hitoshi, und Mikami, Misao. Zur projektiven Theorie der Bahnkurven dritter Ordnung. Proc. Imp. Acad. Tokyo **18**, 595–601 (1942). [MF 14784]

The paper is concerned with the development of a projective theory of paths of the third order. The equations of the paths are

$$d^3x^i/dt^3+H^i(t, x^j, dx^j/dt, d^2x^j/dt^2)=0, \quad i=1, 2, \cdots, n.$$

The authors had previously shown that a projective connection and a covariant differentiation process could be defined in terms of the functions H^i and their partial derivatives up to fourth order. In the present paper it is proved that another projective connection can be constructed if the dimension n of the space exceeds 2. This new connection involves lower order derivatives of the H^i. *A. Fialkow.*

Thomas, T. Y. The fundamental theorem on quadratic first integrals. Proc. Nat. Acad. Sci. U. S. A. **32**, 10–15 (1946). [MF 15312]

Here we are concerned with the set of all homogeneous quadratic first integrals of the equations of paths of an affinely connected space. It is proved that this set admits a finite basis such that any first integral is a linear combination (with constant coefficients) of the integrals forming the basis. The number of integrals in the basis is the same as the number of fundamental sets of solutions of a certain finite set of linear homogeneous algebraic equations. Similar results are believed to hold for homogeneous first integrals of all degrees. Applications of this theorem are made to quadratic first integrals in a flat space and to quadratic first integrals of a conservative dynamical system.

C. B. Allendoerfer (Haverford, Pa.).

Hokari, Shisanji. Einige Sätze über ein System von Pfaffschen Ausdrücken und ihre Anwendungen. Proc. Imp. Acad. Tokyo **17**, 434–443 (1941). [MF 14724]

The author derives various properties of Pfaffian forms in a Kawaguchi space $K_n^{(m)}$. As defined by Kawaguchi, $K_n^{(m)}$ is a manifold each of whose "points" is a parametrized lineal element of order m. Necessary and sufficient conditions that a Pfaffian form defined in $K_n^{(m)}$ transform as a relative tensor with respect to parameter transformations and as a vector with respect to coordinate transformations are found. A differentiation process for these Pfaffian forms is also derived. The work is formal in character.

A. Fialkow (New York, N. Y.).

Hokari, Shisanji. Sätze über ein System von Pfaffschen Ausdrücken in der Mannigfaltigkeit von K-dimensionalen Flächenelementen und ihre Anwendungen. Proc. Imp. Acad. Tokyo **17**, 444–454 (1941). [MF 14725]

It is shown that the results of the paper reviewed above may be extended to apply to Pfaffian forms defined in a space each of whose "points" is a parametrized surface element of order m. *A. Fialkow* (New York, N. Y.).

Ohkubo, Takeo. Die Differentialgeometrie des $(n-1)$-faches Integral. Jap. J. Math. **19**, 33–44 (1944). [MF 14993]

The author studies an n-dimensional manifold in which the volume of a portion of a hypersurface can be defined by an integral of the form

$$\int\int \cdots \int \left\{ A_i^{\alpha\beta}\left(x^j, \frac{\partial x^j}{\partial u^\gamma}\right)\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + B\left(x^j, \frac{\partial x^j}{\partial u^\gamma}\right)\right\}^{1/p} du^1 du^2 \cdots du^{n-1},$$

assumed to be invariant with respect to transformations of the parameters u^α. The question of parallel displacement in this special Cartan space is discussed in detail.

A. Schwartz (State College, Pa.).

Bloch, André. Remarques géométriques sur les principes du calcul tensoriel. C. R. Acad. Sci. Paris **221**, 530–532 (1945). [MF 15140]

The author's object is to revise some geometrical concepts associated with tensor analysis. In particular, he restricts his study to three dimensions and the concepts of contravariant and covariant vectors. [One definition of these quantities is given in Schouten and Struik, Einführung in die Neueren Methoden der Differentialgeometrie, vol. 1, 2d ed., Noordhoff, Groningen-Batavia, 1935]. The author defines a contravariant vector as the combination of a plane and an exterior point, a covariant vector as a vector (in the usual sense). Schemes for the geometric addition and multiplication of n vectors of either type are given. Finally, the author discusses the use of algebraic forms in introducing vectors and tensors. *N. Coburn* (Austin, Tex.).

TOPOLOGY

Gustin, William. Countable connected spaces. Bull. Amer. Math. Soc. **52**, 101–106 (1946). [MF 15447]

The author gives a novel and simplified construction for a countable connected Hausdorff space which has the property that the Urysohn separation axiom fails to hold between each pair of points. An example is also given of a Hausdorff space not satisfying the Urysohn axiom which becomes totally disconnected upon the removal of a single point. *E. Hewitt* (Princeton, N. J.).

Mibu, Yoshimichi. On Baire functions on infinite product spaces. Proc. Imp. Acad. Tokyo **20**, 661–663 (1944). [MF 14940]

Let $E=\prod E_\alpha$ for $\alpha \in E_\alpha$ be the topological product of completely regular spaces E_α; say that a function f defined on E depends on a countable number of coordinates if there exists a countable subset A_f of A such that $f(p)=f(q)$ whenever the αth coordinates of p and q are equal for each α in A_f. Theorems 1, 2 and 3 say that any continuous function on E is determined by a countable number of coordinates if every E_α is, respectively, a closed interval, bicompact and Hausdorff, or completely regular. Theorem 1 is proved by a theorem on the normed ring of continuous functionals on E [Gelfand and Šilov, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 25–39 (1941); these Rev. **3**, 52], theorem 2 by embedding every E_α in a product of unit intervals, and theorem 3 by extending each E_α to a bicompact space so that continuous functions extend uniquely. [Theorem 2 could be proved directly and simply by using the compactness of E.] The definition of the Baire classes gives, from theorem 3,

theorem 4: every Baire function on E is determined by a countable number of coordinates if every E_n is completely regular. If A is uncountable and E_1 is a subset containing just one point of E, E_1 is a closed set whose characteristic function is not a Baire function. *M. M. Day.*

Germain, Paul. Sur le contingent dans certains espaces métriques. C. R. Acad. Sci. Paris **219**, 381–383 (1944). [MF 15268]

For three points m, p, n of a metric space R let $\angle mpn$ denote the angle at p' in a Euclidean triangle $m'p'n'$ isometric with mpn. The space R is "unicontingent" at p if $\angle mpn$ tends to zero for $m \to p$ and $n \to p$. The set $U(p)$ of all subsets of R which are unicontingent at p is divided into classes, two sets belonging to the same class if their union is in $U(p)$. If R is angularly metric at p, that is, if for any three points l, m, n different from p the relation $\angle lpm + \angle mpn \geqq \angle lpn$ holds, then an obvious limit process yields a metrization for the above classes of unicontingent sets. This new metric space U_p^* is complete. Compactness of U_p^* is necessary for a set E with p as accumulation point to have a nonempty contingent. Sufficient conditions are also given. *H. Busemann* (Northampton, Mass.).

Choquet, Gustave. Prolongements d'homéomorphies. Ensembles topologiquement nommables. Caractérisation topologique individuelle des ensembles fermés totalement discontinus. C. R. Acad. Sci. Paris **219**, 542–544 (1944). [MF 15287]

We quote two of seven theorems, all of which are stated without proof. (A) Any homeomorphism between two compact plane subsets of ordinary 3-space can be extended to a homeomorphism of the 3-space onto itself. (B) Any compact homogeneous plane set is either (1) finite, or (2) perfect and totally disconnected or (3) homeomorphic to a union of concentric circles of positive radius which cuts any diameter in a set of type (1) or (2). *R. Arens.*

Choquet, Gustave. Résolution du problème de M. Fréchet sur la paramétrisation d'arcs doués de tangentes. Généralisation aux variétés à plusieurs dimensions. Paramétrages intrinsèques. C. R. Acad. Sci. Paris **221**, 83–86 (1945). [MF 14240]

Soit $\mathbf{V}_n : \mathcal{T}(D)$ l'image d'un domaine D de l'espace euclidien R_n par une transformation continue $\mathcal{T}$; deux telles images $\mathbf{V}_n' : \mathcal{T}'(D')$ et $V_n'' : \mathcal{T}''(D'')$ sont équivalentes si leur écart au sens de Fréchet est nul; $\mathcal{T}'$ et $\mathcal{T}''$ sont alors à regarder comme deux paramétrages de la même $\mathbf{V}_n$. La paramétrisation $\mathcal{T}$ est régulière au point t de D si $\mathcal{T}$ est tangente en t à une transformation linéaire non dégénérée, régulière sans plus si elle est régulière en tout point t de D. Un point $\mathbf{m}$ sur $\mathbf{V}_n$ est non contrariant s'il existe une variété linéaire L_n de R_n contenant le support m de $\mathbf{m}$ telle que, pour tout voisinage W de m dans L_n de diamètre δ, il existe un voisinage $\mathbf{O}$ de $\mathbf{m}$ dans $\mathbf{V}_n$ jouissant de la propriété suivante: ϵ désignant l'écart de Fréchet entre W et $\mathbf{O}$, le rapport $\epsilon : \delta$ tend vers 0 avec δ. Le théorème fondamental contenu dans cette note est le suivant: une condition nécessaire et suffisante pour que $\mathbf{V}_n$ admette une paramétrisation régulière est qu'aucun point de $\mathbf{V}_n$ ne soit contrariant.

Ainsi est résolu pour n quelconque le problème posé par M. Fréchet pour $n = 1$ [Espaces Abstraits, Gauthier-Villars, Paris, 1928, p. 153; Fund. Math. **26**, 334 (1936)]. L'auteur indique qu'il a construit un arc simple A^* ayant une tangente en tout point et pour lequel l'ensemble des points contrariants est dénombrable et dense en soi; or l'ensemble des points où une paramétrisation d'un tel arc n'est pas régulière étant un G_δ, il est impossible de trouver une paramétrisation de A^* qui soit régulière en tous les points non contrariants de A^*. Cet exemple résout négativement la question poseé par le référent dans sa thèse [Les méthodes directes en calcul des variations et en géométrie différentielle, Paris, 1941, pp. 95–96; ces Rev. **7**, 67].

Soit $\mathfrak{A}$ la famille des arcs simples A d'un espace distancié R, métrisés par l'écart de Fréchet; M. Morse [Bull. Amer. Math. Soc. **42**, 915–922 (1936)] a donné une paramétrisation intrinsèque $\mathfrak{M}(t)$ définie sur l'intervalle $I = [0, 1]$ telle que le point $\mathfrak{M}(t) = \mathfrak{M}(A, t)$ soit fonction continue de A et de t dans l'espace $\mathfrak{A} \times I$. L'auteur a montré par un exemple qu'il est impossible, lorsque $R = R_n$, de trouver une paramétrisation intrinsèque pour laquelle $\mathfrak{M}(A, t)$ serait fonction uniformément continue de A et t. *C. Pauc* (Paris).

Puig Adam, P. On the individualization of the directions on closed plane Jordan curves. Revista Mat. Hisp.-Amer. (4) **5**, 204–214 (1945). (Spanish) [MF 15208]

A proof of the possibility of defining the sense of a closed Jordan curve intrinsically, without using the parametrization. *R. Valdes and R. Arens* (Princeton, N. J.).

Koseki, Ken'iti. Über die Begrenzung eines besonderen Gebietes. Proc. Imp. Acad. Tokyo **20**, 406–408 (1944). [MF 14904]

Let k be a subcontinuum of a compact continuum which is the common boundary of 2 plane domains G_1 and G_2. If k is a continuum of condensation of r [or an indecomposable continuum] there is [are] at most one point [two points] of k which is [are] arc-wise accessible from both G_1 and G_2. In each case there are at most two points which are accessible from all sides from G_1. If k is a "tranche fondamentale" [see Kuratowski, Fund. Math. **12**, 20–42 (1928)], then it contains at most a countable number of points which are accessible from all sides from G_1.

J. H. Roberts (Durham, N. C.).

Kondô, Motokiti. Sur la notion de la dimension. Proc. Imp. Acad. Tokyo **19**, 215–223 (1943). [MF 14813]

According to Fréchet, a subset A of a space has dimensional type greater than or equal to that of a subset B if a subset of A is homeomorphic to B. Fréchet's concept is clearly related to that underlying the definition of cardinal number. The present paper extends the generality and abstractness of this concept and applies it to a variety of cases. If $\mathbf{L}$ is a system partially ordered by a relation $\geqq$, having unit and null-element, and if there is defined in $\mathbf{L}$ a symmetric and transitive congruence relation $\cong$, then an element A of $\mathbf{L}$ is said to have dimensional type greater than or equal to that of B, $\dim(A) \geqq \dim(B)$, if there exists an element C such that $A \geqq C$ and $C \cong B$. If both $\dim(A) \geqq \dim(B)$ and conversely, it is said that $\dim(A) = \dim(B)$. It is easily shown that $A \geqq B$ implies $\dim(A) \geqq \dim(B)$ and $A \cong B$ implies $\dim(A) = \dim(B)$. Another congruence postulate, namely, $A \geqq C$ and $A \cong B$ imply the existence of a D such that $B \geqq D$ and $C \cong D$, is needed to insure that $\dim(A) \geqq \dim(B)$ and $\dim(B) \geqq \dim(C)$ imply $\dim(A) \geqq \dim(C)$.

Several postulates are then introduced on orthogonality, denoted by $\perp$, of elements of $\mathbf{L}$ and relations among $\perp$, $\geqq$, $\cup$, $\cong$. The class $\mathbf{D}$ of dimension types is partially ordered, and it is sometimes possible to define the sum of two elements a, b of $\mathbf{D}$ as follows: If $\dim(A) = a$, $\dim(B) = b$, $A \perp B$

and $A \cup B$ exists, then $a+b$ is dim $(A \cup B)$. This sum, if it exists, is unique, commutative, and associative. By a sort of ideal construction, **D** is extended to a **D*** in which addition is always possible.

Following Tarski [Fund. Math. **31**, 47–66 (1938)] an element $[a^*]$ of **D*** is called normal if no integer n exists for which $(n+1)[a^*] \leqq n[a^*]$; $A \varepsilon \mathbf{L}$ is called normal if [dim (A)] is normal. Corresponding to the classification of von Neumann and Murray [Ann. of Math. (2) **37**, 116–229 (1936)], **L** is said to be (α) finite, (β) infinite or (γ) purely infinite according to whether (α) the unit element of **L** is normal, (β) the unit is not normal but there does exist a normal element in **L** or (γ) no element in **L** is normal. For **L** of finite type a construction is given assigning a numerical dimension between 0 and 1 to each element of **L**. For **L** of infinite type a similar construction yields a numerical dimension ranging between 0 and ∞. Dimension in the sense of Hausdorff or of Urysohn and Menger is of purely infinite type.

The paper concludes with particularizations of the general theory to several cases, including those of cardinal numbers, Fréchet dimension type, rings of operators, Hausdorff dimension and Urysohn-Menger dimension. *H. Wallman.*

Wallace, A. D. Extension sets. Trans. Amer. Math. Soc. **59**, 1–13 (1946). [MF 15314]

Proofs and a further discussion of the results announced in another paper [Proc. Nat. Acad. Sci. U. S. A. **31**, 414–417 (1945); these Rev. **7**, 216] concerning a generalization along homotopy lines of the structure theory of continuous curves. *E. G. Begle* (New Haven, Conn.).

Scott, D. B. Intersection groups and rings. Proc. Cambridge Philos. Soc. **42**, 183–184 (1946). [MF 15669]

The author extends a concept defined for an algebraic surface [same Proc. **36**, 414–423 (1940); these Rev. **2**, 137] to more general manifolds. Let $\mathfrak{R}$ be the homology ring of a finite absolute orientable manifold M and $\mathfrak{d}$ a two-sided ideal of $\mathfrak{R}$. Let $\mathfrak{H}^p$ be the pth homology group of M and $\mathfrak{R}^p$ the subgroup of $\mathfrak{H}^p$ composed of the cycles Γ^p for which $\Gamma^p \cdot \Delta = 0$ for every Δ of $\mathfrak{d}$. The p-dimensional intersection group of M relative to $\mathfrak{d}$ is defined to be the quotient group $\mathfrak{T}^p_{\mathfrak{d}} = \mathfrak{H}^p/\mathfrak{R}^p$. These groups are easily combined to give a ring $\mathfrak{R}_{\mathfrak{d}}$, called the intersection ring of M relative to $\mathfrak{d}$. Several examples and a few properties of $\mathfrak{R}_{\mathfrak{d}}$ are discussed. *R. J. Walker* (Ithaca, N. Y.).

Whyburn, G. T. Boundary alternation of monotone mappings. Duke Math. J. **12**, 663–667 (1945). [MF 15509]

The question considered is: under what conditions will a mapping which is monotone on the closure of a plane bounded simply-connected region R be nonalternating on A, the boundary of R? A sufficient condition is that if $x \varepsilon A$ then $f^{-1}f(x)$ intersects every cross cut in R, both regions of which have x on their boundary. (A cross cut is an arc which lies in R except that its end-points are in A.) *J. H. Roberts* (Durham, N. C.).

de Kerékjártó, B. Sur le caractère topologique du groupe homographique de la sphère. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 420–441 (1940). (Hungarian. French summary) [MF 15565]

A French translation appeared in J. Math. Pures Appl. (9) **21**, 67–100 (1942); these Rev. **5**, 60; cf. also Acta Math. **74**, 311–341 (1941); these Rev. **7**, 137.

de Kerékjártó, B. Sur les fondements topologiques de la géométrie projective complexe. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 442–454 (1940). (Hungarian. French summary) [MF 15566]

Dans la première partie de ce mémoire, je démontre, par des moyens élémentaires et très simples, la caractérisation suivante du groupe des homographies et des antihomographies. Si G' est un groupe continu de transformations topologiques de la surface d'une sphère en elle-même tel qu'à deux triples de points quelconques $(A^\circ, B^\circ, C^\circ)$ et (A, B, C) correspondent deux et seulement deux transformations de G' qui changent A° en A, B° en B, C° en C, le groupe G' est homéomorphe au groupe des homographies et des antihomographies d'une variable complexe. Dans la deuxième partie, je déduis par des raisonnements topologiques les théorèmes de M. Élie Cartan concernant la représentation des homographies par des antiinvolutions de première espèce. *Author's summary.*

de Kerékjártó, B. Sur les groupes transitifs de la droite. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 455–475 (1940). (Hungarian. French summary) [MF 15567]

A French translation appeared in Acta Univ. Szeged. Sect. Sci. Math. **10**, 21–35 (1941); these Rev. **2**, 322.

de Kerékjártó, B. Sur les transformations périodiques du plan projectif. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 798–804 (1940). (Hungarian. French summary) [MF 15561]

(I) Toute transformation involutive du plan projectif admet un seul point invariant isolé; ses autres points invariants forment une courbe simple et fermée, isotope à une droite dans le plan projectif. La transformation est homéomorphe à une homologie harmonique. (II) Toute transformation de période $(n>2)$ du plan projectif admet un seul point invariant; il y a une courbe γ isotope à une droite qui est transformée en elle-même par une transformation de periode n ou $n/2$ suivant que n est impair ou pair: si n est pair, la courbe invariante γ est uniquement déterminée. La transformation est homéomorphe à une homologie périodique. *Author's summary.*

de Kerékjártó, B. Sur les groupes compacts de transformations topologiques de la sphère. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 805–828 (1940). (Hungarian. French summary) [MF 15562]

A French translation has already been reviewed [Acta Math. **74**, 129–173 (1941); these Rev. **4**, 3].

Hirsch, Guy. Sur des propriétés de représentations permutables et des généralisations d'un théorème de Borsuk. Ann. Sci. École Norm. Sup. (3) **60**, 113–142 (1943). [MF 14639]

Principal theorem: let T be a continuous mapping of a polyhedron E upon itself which is permutable with homeomorphisms A_i $(E \to E)$ generating a group **A**. Let **B** be the subgroup of **A** which is generated by the A's whose fixed-point sets have nonempty intersections with the fixed-point set of T. Then the Lefschetz number of T is a multiple of the index of **B** in **A**. By taking E to be homologically like an n-sphere and assuming **A** to be finite and its transforma-

tions, except the identity, to be free of fixed points, one obtains theorems of Borsuk [Fund. Math. **20**, 177–190 (1933)], Eilenberg [Duke Math. J. **6**, 428–437 (1940); these Rev. **1**, 319] and the reviewer [Appendix B of Lefschetz, "Algebraic Topology," Amer. Math. Soc. Colloquium Publ., vol. 27, New York, 1942; these Rev. **4**, 84].

P. A. Smith (New York, N. Y.).

Matsumoto, Toshizô. Ueber die Einteilung des Simplexes. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **23**, 245–252 (1941). [MF 15555]

An elementary subdivision of a rectilinear simplicial complex in a Euclidean space so that any simplex s is covered with simplexes of at most half the diameter of s.

S. S. Cairns (Flushing, N. Y.).

NUMERICAL AND GRAPHICAL METHODS

★**Circular and Hyperbolic Functions. Exponential and Sine and Cosine Integrals. Factorial Function and Allied Functions. Hermitian Probability Functions.** British Association for the Advancement of Science. Mathematical Tables, Volume 1. Prepared by the Committee for the Calculation of Mathematical Tables. Second Edition. Cambridge, at the University Press; New York, The Macmillan Company, 1946. xi+72 pp. $2.50.

This is a reprint of the edition of 1931, with an abridged introduction. The table of $x!=\Gamma(x+1)$ has been extended to 18 decimal places.

Corrington, Murlan S. Table of Bessel functions $J_n(1000)$. J. Math. Phys. Mass. Inst. Tech. **24**, 144–147 (1945). [MF 15130]

This table gives eight-decimal values of $J_n(1000)$ for $n=935(1)1035$. A differencing check [by the reviewer] proved the absence of accidental errors in the first six decimals at least. The application of the table is in frequency-modulated radio waves. [Attention may be drawn to the work of J. R. Airey on $J_n(x)$ for x and n nearly equal, Rep. British Assoc. Advancement Sci. **1916**, 59–126, in particular, pp. 92–96].

L. J. Comrie (London).

Neushuler, L. On optimal coalescent tables of squares and cubes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 462–465 (1 plate) (1945). [MF 14422]

In coalescent tables each tabular function is divided into two or more groups, and the complete function is produced by "coalescing" the various components. They are possible when members of the various groups are common to several arguments. In an earlier paper [same C. R. (N.S.) **18**, 259–262 (1938)] the author applied this principle to multiplication tables. He now extends it to powers, in which, as is well known, there is much repetition of terminal figures. The net result is a saving of printing space at the expense of dislocation of successive tabular results and more work on the part of the user.

L. J. Comrie (London).

Sen, D. K. Interpolation and summation formulas and the properties of factorials. II. J. Univ. Bombay (N.S.) **11**, part 5, 3–23 (1943). [MF 13034]

Sen, D. K. Interpolation and summation formulas and the properties of factorials. III. J. Univ. Bombay (N.S.) **12**, part 3, 4–13 (1943). [MF 12940]

Sen, D. K. Interpolation and summation formulas and the properties of factorials. IV. J. Univ. Bombay (N.S.) **12**, part 5, 1–8 (1944). [MF 12943]

Part I has already been reviewed [same J. (N.S.) **11**, part 3, 22–36 (1942); these Rev. **7**, 85]. In part II the author continues his study of the factorial function $x(x-1)\cdots(x-n)=F_n(x)$ with an examination of the mean value of $F_n(x)$ in each subinterval between successive zeros, gives a table for all the means for each value of n from 1 to 10, and determines the asymptotic behavior as $n\to\infty$. He then investigates the three definite integrals

$$M_{2p}=\frac{1}{(2p)!}\int_{p-1}^{p}F_{2p-1}(x)dx,$$

$$K_{2p}=\frac{1}{(2p)!}\int_{-1}^{1}xF_{2p-2}(x+p-1)dx,\quad L_p=\frac{1}{p!}\int_{0}^{1}F_{p-1}(x)dx.$$

The first two are associated with Gauss's summation formulas; the third, with Laplace's summation formula. The author obtains asymptotic expressions and recurrence formulas for these integrals and establishes relations between them.

Part III takes up three expressions introduced by Lubbock and denoted by P_{2p}, Q_{2p}, Λ_p [for explanation of notation see Steffensen, "Interpolation," Williams and Wilkins, Baltimore, 1927, pp. 139 ff.] for which new recurrence formulas and asymptotic expressions are obtained. The summation formulas of Lubbock, Laplace, and Gauss are next investigated with particular attention to their connection with interpolation formulas and their remainder terms.

In part IV the results are specifically applied to interpolation formulas, in particular Newton's, Stirling's, Bessel's, and Everett's. The reviewer was disappointed to find that so elaborate a study added so little to the previous knowledge of the magnitude of the error. The paper concludes with some applications of Newton's interpolation formula to the estimation of population for any age group.

W. E. Milne (Corvallis, Ore.).

Soper, A. K. Interpolation schedule for the Lagrange formula. Nature **157**, 299–300 (1946). [MF 15753]

Vernotte, Pierre. Dérivation des courbes expérimentales. C. R. Acad. Sci. Paris **219**, 579–581 (1944). [MF 15293]

Samuelson, Paul A. A convergent iterative process. J. Math. Phys. Mass. Inst. Tech. **24**, 131–134 (1945). [MF 15127]

The iterative procedure $X_{i+1}=G(X_i)$ is known to converge to the root a of $G(X)-X=0$ if $G'(a)<1$ and the process starts sufficiently near a. The iterative procedure

$$Y_{i+1}=\frac{G_i^{(2)}G_i^{(0)}-G_i^{(1)}G_i^{(1)}}{2G_i^{(1)}-G_i^{(2)}-G_i^{(0)}}=F(Y_i),$$

where $G_i^{(1)}$ is the result of applying G to Y_i i times, converges to a whenever $G'(a)\neq 1$ and the process starts sufficiently near a. When $G'(a)\neq 1$, $F'(a)=0$, so that convergence is more than exponential. Generalizations are discussed.

J. W. Tukey (Princeton, N. J.).

Steffensen, J. F. Further remarks on iteration. Skand. Aktuarietidskr. **28**, 44–55 (1945). [MF 14154]

Let an equation of the form $x=f(x)$ have a positive root $x=X$. If $f(x)$ is decreasing in an interval containing X and if $x_0, x_1, \cdots$ are defined by $x_{n+1}=f(x_n)$, with x_0 sufficiently

close to X, then it is known that X is always contained between x_n and x_{n+1}. The author points out that, if $f(x)$ is increasing in the neighborhood of X, the substitution $x=z^\alpha$ transforms the equation into $z=z^{1-\alpha}f(z^\alpha)=F(z)$, where α can always be chosen so that $F(z)$ is decreasing in a neighborhood of $Z=X^{1/\alpha}$. This device is illustrated by a number of examples in which rates of interest are computed by iterations. *Z. W. Birnbaum* (Seattle, Wash.).

Holubář, Josef. Graphical solution of equations of 2d, 3d and 4th degrees. Rozhledy Matematicko-Přírodovědecké 21, 11–16 (1941). (Czech) [MF 15693]

Ruggiero, R. J. Investigation of three methods for solving the flutter equations and their relative merits. J. Aeronaut. Sci. 13, 3–22 (1946). [MF 15132]

A comparison of the time required to find the characteristic roots and vectors of a matrix by simple iterations or by direct expansion. No modern refinements of techniques are used. The matrices considered are not of high order, and no great accuracy is required. A substantial part of the paper is expository and serves as an introduction to the algebra of matrices. *W. Feller* (Ithaca, N. Y.).

Berry, Clifford E. A criterion of convergence for the classical iterative method of solving linear simultaneous equations. Ann. Math. Statistics 16, 398–400 (1945). [MF 15475]

The equations in matrix form are $AX=G$, $A=A_1+A_2$, where A_1 is zero above the diagonal and A_2 is zero on and below the diagonal. The classical iteration converges if and only if $(A_1^{-1}A_2)^m$ approaches zero as $m\to\infty$. Simpler sufficient criteria derived from this are shown to be accessible to easy calculation. *J. W. Tukey* (Princeton, N. J.).

Dwyer, Paul S. The square root method and its use in correlation and regression. J. Amer. Statist. Assoc. 40, 493–503 (1945). [MF 15125]

The author has recently developed considerable simplifications of the computational techniques connected with correlation analysis [Ann. Math. Statistics 15, 82–89 (1944); these Rev. 5, 245]. "It is the aim of this paper to present in some detail, and with illustrations to a correlation problem previously used by the author in discussing compact techniques [J. Amer. Statist. Assoc. 37, 441–460 (1942); these Rev. 4, 164], the 'square root' method of solving equations." The latter method forms part of the paper first quoted. *W. Feller* (Ithaca, N. Y.).

Zimmerman, Wayne S. A simple graphical method for orthogonal rotation of axes. Psychometrika 11, 51–55 (1946). [MF 15677]

van Veen, S. C. Stark konvergente Entwicklungen für die vollständigen elliptischen Integrale erster und zweiter Art. Nederl. Akad. Wetensch., Proc. 44, 964–973, 1077–1084, 1198–1205 (1941); 45, 32–36 (1942). [MF 14304]

The usual hypergeometric series for the complete elliptic integrals,

$$K(k)=\int_0^{\pi/2}(1-k^2\sin^2\varphi)^{-\frac{1}{2}}d\varphi=\tfrac{1}{2}\pi F(\tfrac{1}{2},\tfrac{1}{2};1;k^2),$$

$$E(k)=\int_0^{\pi/2}(1-k^2\sin^2\varphi)^{\frac{1}{2}}d\varphi=\tfrac{1}{2}\pi F(-\tfrac{1}{2},\tfrac{1}{2};1;k^2),\qquad 0\leqq k<1,$$

converge satisfactorily only for small values of k. The author obtains much more rapidly convergent series by a method of transformation of parameters, first for values of k not too near unity and then for k very nearly equal to unity. In the first case two sets of auxiliary parameters a_n, b_n are introduced,

$$a_0=1+k,\quad b_0=1-k,\quad a_n=\tfrac{1}{2}(a_{n-1}+b_{n-1}),\quad b_n=(a_{n-1}b_{n-1})^{\frac{1}{2}},$$

and then a set $k_n=(a_{n-1}-b_{n-1})/(a_{n-1}+b_{n-1})$, with $k_1=k$. The author next shows that

$$K(k)=a_1^{-1}K(k_1)=\cdots=a_n^{-1}K(k_n)=\tfrac{1}{2}\pi a_n^{-1}F(\tfrac{1}{2},\tfrac{1}{2};1;k_n^2).$$

Since k_n decreases rapidly with increasing n, the successive hypergeometric series are more and more closely approximated by their leading terms. For $E(k)$ a similar though somewhat more complicated expression is obtained. When k is very nearly unity the original hypergeometric functions are first transformed to power series in $(1-k^2)$ and then a complementary set of parameters $l_n(\alpha)=k_n(\frac{1}{2}\pi-\alpha)$, $k=\sin\alpha$, is introduced.

The author writes out the various expansions in detail for the first few values of n. In a final section he discusses the convergence of the various series for complex values of k and shows that the expansions are valid in the right half plane, $\Re(k)\geqq 0$, with a cut along the real axis from $+1$ to $+\infty$. *M. C. Gray* (New York, N. Y.).

van Veen, S. C. Ueber die Entwicklung der unvollständigen elliptischen Integrale erster und zweiter Art in stark konvergenten Reihen. Nederl. Akad. Wetensch., Proc. 44, 974–977, 1085–1091, 1206–1209 (1941); 45, 37–42 (1942). [MF 14303]

This is a companion paper to the one reviewed above. For the incomplete elliptic integrals

$$F(\sin\alpha,\beta)=\int_0^\beta(1-\sin^2\alpha\sin^2\varphi)^{-\frac{1}{2}}d\varphi,$$

$$E(\sin\alpha,\beta)=\int_0^\beta(1-\sin^2\alpha\sin^2\varphi)^{\frac{1}{2}}d\varphi,$$

the author considers three distinct sets of values of α and β. (1) α not near $\frac{1}{2}\pi$, β arbitrary. In this case the parameters of the preceding paper are used, with an additional angular parameter $\beta_n=\arctan\{\sin 2\beta_{n-1}/(k_n+\cos 2\beta_{n-1})\}$. The series for $n=3$ are given in detail. For example, the leading term for $F(\sin\alpha,\beta)$ is

$$(1+(\cos\alpha)^{\frac{1}{2}})^{-2}\{\beta+\arctan(\cos\alpha\tan\beta)+2\arctan((\cos\alpha)^{\frac{1}{2}}\tan\beta)\}.$$

(2) α near $\frac{1}{2}\pi$, β small. Elementary transformations of the integrals lead to expansions in power series with variable $(1-\sin\alpha)/(\sin\alpha)^{\frac{1}{2}}$. There is, of course, a logarithmic factor, the leading term for $F(\sin\alpha,\beta)$ being

$$\tfrac{1}{2}(\sin\alpha)^{-\frac{1}{2}}\log\{(1+\sin\beta(\sin\alpha)^{\frac{1}{2}})/(1-\sin\beta(\sin\alpha)^{\frac{1}{2}})\}.$$

(3) α and β both near $\frac{1}{2}\pi$. Expansions similar to those of (2) are obtained for the functions representing the difference between the corresponding complete and incomplete integrals. *M. C. Gray* (New York, N. Y.).

Mikheladze, Š. On the numerical integration of a function depending on a parameter. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 5, 575–583 (1944). (Georgian. Russian summary) [MF 14607]

By means of the method of least squares, approximation formulae are obtained for the computation of the values of integrals with variable limits of integration. The method is

extended to iterated integrals and to integrals depending on a parameter, and is applied, by way of example, to an elliptic integral. *E. F. Beckenbach* (Los Angeles, Calif.).

Nyström, E. J. Zur praktischen Integration von linearen Differentialgleichungen. Soc. Sci. Fenn. Comment. Phys.-Math. **11**, no. 14, 14 pp. (1943). [MF 14632]

The purpose of the paper is to obtain a numerical solution of the boundary value problem

$$d^2z/dx^2+\alpha(x)z+\beta(x)=0, \qquad z(x_a)=z_a,\ z(x_b)=z_b.$$

A linear transformation carries this into the boundary value problem with fixed interval $(-\frac{1}{2}, \frac{1}{2})$ and homogeneous boundary conditions. This system is then replaced by an equivalent integral equation. To solve the integral equation, the unknown function under the integral sign is replaced by a Lagrange polynomial of fourth degree involving five equally spaced ordinates (three interior to the interval and the two zero end ordinates). Three simultaneous linear equations containing the three interior ordinates result from the integration, and the solution of these equations leads to an approximate solution of the original problem. The accuracy can be increased by using not equally spaced ordinates but ordinates taken at certain irrationally spaced points in analogy with Gauss's method of integration. Three illustrative examples are worked, and the paper concludes with a summary of the essential formulas. *W. E. Milne.*

Pimiä, Lauri. Interpolationsverfahren zur Berechnung der Flugbahnen einer Schar sowie ihrer Veränderung durch Variation des Abgangswinkels. Soc. Sci. Fenn. Comment. Phys.-Math. **12**, no. 7, 13 pp. (1944). [MF 14627]

Consider a one-parameter system of trajectories of a projectile with constant ballistic coefficient and constant initial velocity, but with variable angle of departure φ. The author investigates interpolation with respect to φ based on a small number of calculated trajectories. The trajectory is referred to oblique coordinates (ξ, η) where the origin is at the muzzle O; $O\xi$ lies along the initial direction of flight and $O\eta$ is directed vertically downward. The coordinates of position (ξ, η) and of velocity $(\dot{\xi}, \dot{\eta})$ are expanded in power series in the parameter $\sin\varphi$ with coefficients which are functions of t alone. Suppose trajectories are known for three values of φ. Then the power series are terminated after three terms and appropriate interpolation formulas are obtained. An examination of the error term throws light on the best choices of the φ's for the three fundamental trajectories. The author finds these to be $\varphi_1=3°50'28''$, $\varphi_2=30°$, $\varphi_3=68°54'34''$.

Several numerical examples illustrate the procedure in computing interpolated values for ξ, η, $\dot{\xi}$, $\dot{\eta}$. The rates of variation of ξ, η, $\dot{\xi}$ and $\dot{\eta}$ with respect to φ are also readily obtained from the same developments. *W. E. Milne.*

Liikkanen, Ilmari. Zur Störungstheorie der äusseren Ballistik. Soc. Sci. Fenn. Comment. Phys.-Math. **12**, no. 1, 72 pp. (1943). [MF 15371]

This is a thesis on mathematical methods in ballistics, presenting a survey of the mathematical theory with historical and bibliographical references, including a treatment of weighting factors (based on equal time-intervals along the trajectory). No suggestion is made of the dependence of the drag function upon the projectile characteristics other than through a single coefficient, the form factor. No new theorems are found. No mention is made of Bliss's method. *A. A. Bennett* (Providence, R. I.).

Alt, Franz L. Multiplication of matrices. Mathematical Tables and Other Aids to Computation **2**, 12–13 (1946). [MF 14670]

The I. B. M. (or Hollerith) multiplying punch has been in use since 1932 for scientific work. Alt describes a new and more powerful multiplier, still in a semi-experimental stage, that is being "proved" at the Ballistics Research Laboratory of the Aberdeen Proving Ground, Maryland. Although it is more or less a general-purpose calculator, the particular application described is the multiplication of two matrices. From cards in which each of the various elements and their row and column numbers are punched, the machine, without any attention on the part of the operator, will complete the entire multiplication and record the desired answers. Its speed is such that two 9-row matrices with elements of not more than six significant figures can be multiplied in nine minutes. *L. J. Comrie* (London).

Bachmann, W. K. Résolution mécano-optique d'un système de deux équations linéaires à deux inconnues. Schweiz. Z. Vermessgswes. Kulturtech. **40**, 241–247 (1942). [MF 15571]

Bush, V., and Caldwell, S. H. A new type of differential analyzer. J. Franklin Inst. **240**, 255–326 (1945). [MF 13620]

This paper contains a description of the new differential analyzer which was developed and is now in use at the Massachusetts Institute of Technology. In fundamental principle it is like previous differential analyzers. Variables in the differential equation to be solved are represented by positions of rotating shafts. These are interconnected by various elements in such a way as to enforce the relations implied by the equation. These elements are of four types: (1) integrators which give $W=\int U dV$, (2) adders which give $W=\pm U\pm V$, (3) gear boxes which give $U=KV$, and (4) function units producing $U=f(V)$. The new machine has greatly increased capacity, speed, flexibility and precision. It contains 18 integrators and 16 adders. The interconnection is done electrically by means of servomechanisms and an electrical switching system allowing rapid setting-up and change of problems. Problems are introduced in the machine by three punched tapes; the A tape sets up the proper interconnection of elements for the equation, the B tape sets gear boxes to the proper values, and the C tape sets in the boundary conditions (initial settings of the integrators). Up to three equations can be solved simultaneously, using different parts of the machine. The precision is of the order of 1 part in 10000, depending on the type of equation. In the paper the general method of setting up equations is indicated and the many improvements over previous analyzers are described. *C. E. Shannon* (New York, N. Y.).

Bachmann, W. K. Note sur la théorie générale des planimètres. Schweiz. Z. Vermessgswes. Kulturtech. **41**, 36–38 (1943). [MF 15572]

Bachmann, W. K. Calcul du relèvement par inversion. Schweiz. Z. Vermessgswes. Kulturtech. **39**, 205–208, 221–226 (1941). [MF 15569]

The author applies a transformation by inversion to the "three point problem" of plane surveying. Formulae for a numerical solution are derived and interpreted geometrically. *E. Lukacs* (Cincinnati, Ohio).

Leemann, W. Über eine praktische Anwendung der Regula Falsi. Schweiz. Z. Vermessgswes. Kulturtech. **41**, 276–278 (1943). [MF 15577]

The problem is to find a circular arc having given tangents at its ends and equal distances from two given points.

Ansermet, A. La solution dite numérique du problème fondamental de la photogrammétrie. Schweiz. Z. Vermessgswes. Kulturtech. **41**, 169–173 (1943). [MF 15573]

Bachmann, W. Ch. Note sur l'article "La solution dite numérique du problème fondamental de la photogrammétrie." Schweiz. Z. Vermessgswes. Kulturtech. **41**, 235–238 (1943). [MF 15576]

Tichonoff, A. N., and Boulanger, J. D. On the averaging of gravimetric fields. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **9**, 240–260 (1945). (Russian. English summary) [MF 15352]

In general, a gravity map is the result of superposition of two phenomena: large "regional" gravitational anomaly and local anomalies which are the object of study. To interpret them in terms of structural geology, the regional anomaly must first of all be subtracted; otherwise the local anomalies remain unknown. This correction of a gravity map for regional anomaly is a very difficult problem, inasmuch as it must be deduced from the map which it is supposed to correct. This article deals with an old, well-known and very inaccurate, though frequently used, method of averaging the mapped values within a certain arbitrarily chosen area A and considering the average as the regional value at the center of A. Shifting this area through the map, one is supposed to get the picture of regional anomaly. The authors try to improve this basically false method and give some rules for choosing the relative size of the arbitrary area A. In the opinion of the reviewer, their work is useless; a better approximation to the regional anomaly as represented by a quadric is obtained by the method of least squares applied to the whole map. *E. Kogbetliantz* (New York, N. Y.).

Zwinggi, Ernst. Ein Multiplikationssatz für das Deckungskapital. Näherungsweise Berechnung der Versicherungswerte für verbundene Leben. Mitt. Verein. Schweiz. Versich.-Math. **45**, 375–383 (1945). [MF 15425]

Let ${}_tV_{xy:\overline{n}|}$ be the policy value (premium reserve) at the end of t years of an endowment assurance of duration n on the joint lives x and y. Denote by ${}_tV_{x:\overline{n}|}$, ${}_tV_{y:\overline{n}|}$, ${}_tV_{\overline{n}|}$ the corresponding policy values for the single lives and the capital redemption insurances, respectively. It is shown that

$$1-{}_tV_{xy:\overline{n}|}\sim(1-{}_tV_{x:\overline{n}|})(1-{}_tV_{y:\overline{n}|})/(1-{}_tV_{\overline{n}|}).$$

This approximation is derived from Lidstone's formula [J. Inst. Actuaries **33**, 354–356 (1898)] using the differential equation of the premium reserve. A similar formula is also given for the last survivorship endowment assurance.

E. Lukacs (Cincinnati, Ohio).

Leepin, Peter. Das Zinsfussproblem bei der temporären Leibrente als praktische Aufgabe. Mitt. Verein. Schweiz. Versich.-Math. **45**, 289–310 (1945). [MF 15423]

MECHANICS

Kubota, Tadahiko. Einige Bemerkungen zur Kinematik. Tôhoku Math. J. **49**, 106–111 (1942). [MF 14700]

The author develops some formulae and theorems about 1-parameter families of motions in the plane or in space: the radii of curvature of the fixed and moving centrode, Savary's formula, Resal's theorem, a space analogue of the latter, given a disk turning around O, how to find another disk turning around O_1 such that the two disks always touch but do not slide along each other, and a space analogue of this. *H. Samelson* (Syracuse, N. Y.).

Bloch, A. Systems with gyroscopic coupling terms. Philos. Mag. (7) **36**, 440–441 (1945). [MF 15482]

Comment on communications by the author and by H. Jefferson [Philos. Mag. (7) **35**, 315–334 (1944); **36**, 223–224 (1945); these Rev. **6**, 23; **7**, 90].

Wolkowitsch, David. Sur la géométrie des masses et son application à la géométrie des coniques et aux intégrales multiples. C. R. Acad. Sci. Paris **219**, 569–571 (1944). [MF 15289]

Statements of some theorems concerning two conics associated with the dynamics of a system of mass points in a plane. The conics are ellipses for positive masses, but may be hyperbolas if negative masses are admitted.

P. Franklin (Cambridge, Mass.).

Četajev, N. G. Concerning a problem of Cauchy. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 139–142 (1945). (Russian. English summary) [MF 14039]

The author considers a stable motion of a holonomic mechanical system and "infinitesimal perturbations" of that motion. The discussion is based to a large extent on the fundamental memoir of Liapounoff of 1892 [French translation in Ann. Fac. Sci. Toulouse (2) **9**, 203–474 (1907)]. The perturbations are considered as forming a group of motions relative to the stable motion and it is shown that this group is unimodular and possesses a definite quadratic form as an invariant. *G. Y. Rainich.*

Tedone, G. Qualche applicazione di una proprietà di media dello stress. Boll. Un. Mat. Ital. (2) **4**, 93–99 (1942). [MF 16060]

Si effettua l'applicazione di una proprietà di media dello stress al caso di un solido pesante girevole senza attrito attorno a un asse fisso e in specie al pendolo composto.

Author's summary.

Bloch, Z. S. Synthesis of mechanisms for motions slightly deviating from uniform motion. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 492–494 (1945). (Russian. English summary) [MF 15443]

Astronomy

Brouwer, Dirk. The motion of a particle with negligible mass under the gravitational attraction of a spheroid. Astr. J. **51**, 223–231 (1946). [MF 15436]

This paper treats the problem of the motion of a particle of negligible mass under the gravitational attraction of a nonhomogeneous spheroid. The method used is similar to that used by G. W. Hill and later by Brown in the lunar theory. Deviations from a known circular periodic orbit are

discussed. In the plane of the equator of the spheroid the solutions are obtained to the third power of the eccentricity. The motion outside the equatorial plane is also considered and use is made of G. W. Hill's infinite determinant used in the motion of the lunar perigee. Finally, an application is made to the motion of the fifth satellite of Jupiter but the author does not claim very exact confirmation of his theory by the observations. It is not shown that the solutions in the equatorial plane converge. *H. E. Buchanan.*

de Jekhowsky, Benjamin. Sur la suppression des approximations successives dans la méthode de Lagrange-Andoyer de détermination des orbites paraboliques. C. R. Acad. Sci. Paris **219**, 605–607 (1944). [MF 15297]

Taalimi, Mohamed. Étude sur la probabilité des comètes à orbite hyperbolique. Arch. Sci. Phys. Nat., Geneva **22**, 32–62, 105–146 (1940). [MF 14188]

A detailed account is given of earlier work on the probability of elliptic and hyperbolic orbits of comets. The analysis by G. Tiercy [Comment. Math. Helv. **4**, 195–218 (1932)] is reproduced in detail and further developed. For the velocity distribution at a distance from the sun exceeding 40,000 astronomical units, the expression $\lambda(v)=\exp\{-k(v^m-U^m)^{2p}\}$ is adopted. With the aid of numerical integrations applications are made for two different sets of constants k, m, U, p. It is shown that a velocity distribution of the adopted form may produce the observed large ratio between elliptic and hyperbolic orbits. [It appears to the reviewer that the author's conclusion in favor of a theory of interstellar origin of comets is not sufficiently supported by his analysis.]

D. Brouwer (New Haven, Conn.).

Järnefelt, G. Zum Einkörperproblem in dem sich ausdehnenden Raume. Ann. Acad. Sci. Fennicae. Ser. A. **55**, no. 3, 21 pp. (1940). [MF 14654]

Following McVittie [Monthly Not. Roy. Astr. Soc. **93**, 325–339 (1933)], the author studies the influence of the expansion of the universe on planetary motion. The Einstein-de Sitter universe is adopted as the cosmological background. In this model, observations on the nebular red shift indicate that the present age of the universe is 1.2×10^9 years, and that the universe has expanded by a factor 1.51 in this period. The expansion factor for a time interval of 10^{15} years is 1.78×10^4. During this period of great expansion the motion of the earth about the sun remains practically identical with that given by Newtonian theory. It is also shown that the earth's orbit remains within fixed and narrow bounds at all times. Planetary motion does not share in the expansion of the universe. The author outlines how his arguments can be applied to other expanding cosmological models with Euclidean space. *A. Schild.*

Kopal, Zdeněk. An outline of the theory of atmospheric eclipses. Proc. Amer. Philos. Soc. **89**, 590–600 (1945). [MF 14115]

The paper presents a theoretical analysis of the absorption effect on the light curve of an eclipsing binary system if one of the two components (or both) possesses an extended atmosphere. *D. Brouwer* (New Haven, Conn.).

Tiercy, Georges. Les binaires à éclipses, la vitesse de la lumière et les théories relativistes. Arch. Sci. Phys. Nat., Geneva **23**, 5–24 (1941). [MF 14190]

The author expresses his preference for a relativistic theory which preserves classical space and absolute time, the propagation of light being nonisotropic. This theory is stated to be conceptually simpler than Einstein's special theory of relativity and free of the inner contradictions which are attributed to the latter. The Doppler effect of double stars is examined, first on P. Dive's hypothesis [Bull. Astr. (2) **12**, no. 1 (1940)] of nonisotropic light propagation with the source of radiation in the centre of ellipsoidal waves; then on J. Le Roux's hypothesis [Relativité Restreinte, Gauthier-Villars, Paris, 1922] of ellipsoidal waves focalized on the source of radiation. In both cases the Doppler effect agrees with the relativistic effect to the first order in v/c. *A. Schild* (Toronto, Ont.).

Tiercy, Georges. À propos de la conclusion qu'on peut tirer du problème des étoiles doubles en ce qui concerne les théories relativistes. Arch. Sci. Phys. Nat., Geneva **24**, 25–36 (1942). [MF 14193]

Assuming P. Dive's hypothesis [see the preceding review] of nonisotropic light propagation, the author examines the possibility of observing the satellite of a double star system at two points of its orbit (about 90° apart) simultaneously. The calculated distance Δ at which such an effect could be observed is, in the case of known visual double stars, from 10^8 to 10^{13} lightyears. These distances are large compared to the dimensions of our galaxy. The author concludes that the known astronomical observations on visual double stars are not in disagreement with Dive's hypothesis.

A. Schild (Toronto, Ont.).

Tiercy, Georges. La signification du problème des étoiles doubles pour les théories relativistes. C. R. Séances Soc. Phys. Hist. Nat. Genève **59**, 45–47 (1942). [MF 14196]

The author considers spectroscopic double stars and finds that Δ [see the preceding review] is of the order of 10^4 lightyears. At this distance the components of the double star system cannot be resolved by our telescopes.

A. Schild (Toronto, Ont.).

Mayot, Marcel. Sur la forme et les mouvements internes d'amas d'étoiles dont le centre de gravité décrit une courbe quelconque. C. R. Acad. Sci. Paris **218**, 827–828 (1944). [MF 15330]

Coutrez, Raymond. Sur la dynamique de la voie lactée. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 660–675 (1942). [MF 13674]

The paper deals with the well-known problem of deriving the consequences for dynamical theory implied by the known kinematics of stellar motions. [The author does not seem to be acquainted with the extensive literature which already exists on this subject.] *S. Chandrasekhar.*

Järnefelt, G. Zur rechnerischen Ausführung eines Sternmodells mit konstanter Energieerzeugung. Soc. Sci. Fenn. Comment. Phys.-Math. **12**, no. 5, 31 pp. (1944). [MF 14626]

The paper considers in detail the equations of stellar equilibrium for the case when the energy sources are distributed uniformly through the star and the law of opacity is that of Kramers-Eddington. The solution of the equations is considered in two steps. First, an equation involving the temperature and density alone is derived and solutions of this equation satisfying the necessary boundary conditions of the problem are obtained by a method of successive substitutions. The method is rapidly convergent and appears most suitable for the problem. Next, the derived temperature-density relations are used in the remaining equations to determine the march of the variables through

the star. Numerical examples illustrating the method are provided. *S. Chandrasekhar* (Williams Bay, Wis.).

Baumgartner, Willy. Zur Verflüssigung von Gaskugeln. Helvetica Phys. Acta 18, 167–194 (1945).

The central problem considered in this paper is the equilibrium under its own gravitation of a mass of vapor in thermodynamic equilibrium with its liquid phase. An integrated form of the Clapeyron-Clausius equation is used in conjunction with the equations of hydrostatic gravitational equilibrium to investigate the structure of such configurations. It is shown that, with a value for the heat of vaporization corresponding to that of iron, masses of planetary order are obtained interior to points where the temperature exceeds 1300° K. It is noted in this connection that the melting point of iron is 1530° K. However, the radii of the resulting configurations are of the order of 10^{12} to 10^{13} cm. Nevertheless, the author advances arguments to support his view that configurations of the type considered are relevant to the problem of planetary constitution during the early stages of their formation. *S. Chandrasekhar.*

Hagihara, Yusuke. The electron velocity distribution in the celestial gaseous assemblies in radiative equilibrium. Proc. Imp. Acad. Tokyo 20, 493–500 (1944). [MF 14914]

The author gives a brief account of methods and results of eight former papers in Jap. J. Astr. Geophysics [1940–1943] on the radiation mechanism in planetary nebulae. As these are not in thermodynamic equilibrium, there is no detailed balancing between each transition and its reverse. Instead, the steady state is characterized by equations for cyclic transitions for each kind of particle: electrons, protons, H-atoms and doubly ionized oxygen atoms. The electron velocity distribution is developed into a Hermite series. The unknowns are the electron temperature T_e, a coefficient β_2 of the series, the concentrations of the different particles and the temperature T_* of the diluted black body radiation from the exciting star. The data are the observed spectral intensities. The effective cross sections for the different collisions are known from quantum-mechanical calculations. Results are given for five nebulae, T_* ranging from 5000 to 10000, T_e from 30000 to 60000, the last in fair agreement with observations of another kind.

F. Zernike (Groningen).

Hydrodynamics, Aerodynamics, Acoustics

Lichnerowicz, André. Sur les équations de l'hydrodynamique des fluides visqueux et la notion de fluide incompressible en relativité générale. C. R. Acad. Sci. Paris 219, 270–272 (1944). [MF 15255]

For a viscous fluid in general relativity, the author defines an energy tensor

$$(1)\qquad T_{\alpha\beta}=(\rho+p)u_\alpha u_\beta-(p-\lambda F^2\overline{\nabla}_\rho\bar{C}^\rho)g_{\alpha\beta}+2\mu\gamma_{\alpha\beta},$$

where ρ is the density, p the pressure, u_α the unit velocity vector, $g_{\alpha\beta}$ the metric tensor of space-time, $F=\exp\int_{p_0}^{p}(\rho+p)^{-1}dp$, $\bar{C}^\rho=F^{-1}u^\rho$, $\overline{\nabla}_\rho$ is the operator of covariant differentiation with respect to the Eisenhart metric $\overline{ds}^2=F^2ds^2$, $\gamma_{\alpha\beta}$ is the rate-of-deformation tensor $\frac{1}{2}(\overline{\nabla}_\alpha C_\beta+\overline{\nabla}_\beta C_\alpha)$, $C_\alpha=Fu_\alpha$, and λ, μ are two scalar constants characteristic of the fluid. The conservation equation gives

$$(2)\quad (\rho+p)u^\alpha\nabla_\alpha u_\beta=(g_\beta{}^\alpha-u^\alpha u_\beta)\{\partial_\alpha p-\lambda\partial_\alpha(F^2\overline{\nabla}_\rho\bar{C}^\rho)-2\mu\nabla_\rho\gamma_\alpha{}^\rho\}.$$

Since a (p,ρ) relation is implied by the existence of F, that relation with (2) and the condition that u_α is a unit vector give 6 equations for the 6 dependent variables ρ, p, u_α. Equation (2) is the generalization of the Navier-Stokes equations for a viscous fluid. If $\lambda=0$ and the fluid is incompressible ($\nabla_\rho C^\rho=0$) and if, furthermore, the motion is irrotational ($\nabla_\alpha C_\beta-\nabla_\beta C_\alpha=0$), then (2) reduces to $\bar{C}^\alpha\overline{\nabla}_\alpha C_\beta=0$; this implies that the stream lines are geodesics for the Eisenhart metric. This is a generalization of the classical theorem that irrotational motions of a viscous fluid coincide with motions of a perfect fluid. *J. L. Synge.*

Sakadi, Zyuro. On the extension of the differential equations of incompressible viscous fluid. II. Proc. Phys.-Math. Soc. Japan (3) 24, 719–722 (1942). [MF 15041]

[Part I appeared in the same Proc. (3) 23, 27–33 (1941); these Rev. 2, 328.] Notes on the subject of nonlinear stress rate of strain relations. *E. Reissner* (Cambridge, Mass.).

Poritsky, H. Compressible flows obtainable from two-dimensional flows through the addition of a constant normal velocity. J. Appl. Mech. 13, A-61–A-65 (1946). [MF 15738]

The author proves that, by adding a constant velocity $V=v_*$ in the z-direction to a two-dimensional flow in the (x, y)-plane, the resultant three-dimensional velocity field satisfies the exact fluid-flow equations. For the viscous fluid, there is, of course, the difficulty at the solid boundary. Examples of this type of flow obtained by superimposing the constant velocity V on two-dimensional nonviscous compressible flows such as vortex-source, vortex-sink and Prandtl-Meyer flow around a corner are discussed.

H. S. Tsien (Pasadena, Calif.).

Pérès, Joseph. Quelques transformations des équations du mouvement d'un fluide compressible. C. R. Acad. Sci. Paris 219, 501–504 (1944). [MF 15281]

The author states that, if the relations between $\varphi(q,\theta)$ and $\psi(q,\theta)$ are $h\partial\varphi/\partial\theta=\partial\psi/\partial q$, $k\partial\psi/\partial\theta=-\partial\varphi/\partial q$, where h and k are functions of q only, and if $d\mu/dq=h-k\mu^2$, $h=\alpha\mu$, $k=\beta/\mu$, then the new functions $\varphi_1(q,\theta)$, $\psi_1(q,\theta)$ defined by

$$\varphi_1=\beta^{-1}\partial\varphi/\partial q-\varphi=-\mu^{-1}\partial\psi/\partial\theta-\varphi,$$
$$\psi_1=\alpha^{-1}\partial\psi/\partial q-\psi=\mu\partial\varphi/\partial\theta-\psi$$

satisfy the relations $h_1\partial\varphi_1/\partial\theta=\partial\psi_1/\partial q$, $k_1\partial\psi_1/\partial\theta=-\partial\varphi_1/\partial\theta$, where $h_1=\beta\mu=k\mu^2$, $k_1=\alpha/\mu=h/\mu^2$. This transformation is then applied to the two-dimensional irrotational flow of compressible nonviscous fluid where the pressure is a function of density only. One of the many possible transformations leads to the Kármán-Tsien method [J. Aeronaut. Sci. 6, 399–407 (1939)] of approximate solution for subsonic flow. *H. S. Tsien* (Pasadena, Calif.).

Bergman, Stefan. On two-dimensional flows of compressible fluids. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 972, 81 pp. (3 plates) (1945). [MF 14401]

In two-dimensional irrotational motion of inviscid fluid we distinguish three planes: the physical or (x, y)-plane, the hodograph or $(u, -v)$-plane and the logarithmic or $(\lambda, -\theta)$-plane. Here u, v are velocity components and θ is the angle between the velocity and the x-axis in the physical plane. When the fluid is incompressible, the Cartesian coordinates in the logarithmic plane are $\lambda=\frac{1}{2}\log(u^2+v^2)$, $-\theta$. Given an arbitrary analytic function $f(\zeta)$, $\zeta=\lambda-i\theta$, the imaginary part of $f(\zeta)$ determines the stream function (in the logarithmic plane) of a possible incompressible flow; this stream function can be interpreted in the hodograph plane and then in the physical plane, there yielding actual streamlines. When the fluid is compressible, λ is replaced by a

certain logarithmic function $\lambda(M)$ of M, the local Mach number, and the author shows that the arbitrary $f(\zeta)$, $\zeta=\lambda(M)-i\theta$, now determines a function of ζ whose imaginary part yields the stream function of a subsonic compressible flow, which may be interpreted similarly in the physical plane. Conditions are obtained that the flow pattern will represent flow past a closed curve. The main aerodynamical problem of determining the flow past a given profile appears to be solvable only by successive approximations. The author considers methods of solution and also the mixed problem where the flow is partially supersonic. The success of the hodograph method depends fundamentally on the fact that the stream function, in the hodograph and logarithmic planes, satisfies a linear equation.

L. M. Milne-Thomson (Greenwich).

Bergman, Stefan. Graphical and analytical methods for the determination of a flow of a compressible fluid around an obstacle. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 973, 29 pp. (8 plates) (1945). [MF 14400]

[See the preceding review.] The formulae previously obtained are applied to find the flow past an obstacle. The actual computation of the stream function generated by a given analytic function is described in detail and methods are discussed for choosing the function which leads to compressible flow past a shape similar to a given obstacle. The procedure is illustrated by carrying out completely the construction of the compressible flow generated by the analytic function whose imaginary part yields, for an incompressible fluid, the flow, without circulation, past a circular cylinder. The corresponding compressible flow is found to take place round a cylinder of somewhat distorted cross-section. The diagram is given in the case $k=-0.5$ ($p=A+\sigma\rho^k$) and the author promises a diagram for the more interesting case $k=1.4$ in a future paper. *L. M. Milne-Thomson.*

Couchet, Gérard. Sur les mouvements plans, non stationnaires, à circulation constante. C. R. Acad. Sci. Paris 221, 280–282 (1945). [MF 14499]

The nonuniform two-dimensional motion of an airfoil, with the circulation determined by the Kutta-Joukowski condition, requires in general that the circulation is a function of time and that consequently a surface of discontinuity is left behind. The author indicates that nonuniform motions are possible with constant circulation and gives the condition for this to be the case. Formulas for lift and moment which apply under this condition are also given.

E. Reissner (Cambridge, Mass.).

Kawamura, Tōru. A study on the subsonic flow of a compressible fluid past a circular cylinder. Proc. Phys.-Math. Soc. Japan (3) 24, 110–119 (1942). [MF 15022]

The stream function for the subsonic compressible flow past a circular cylinder is calculated to the order M^4, where M is the Mach number of the main flow, in order to study the streamlines of the flow. The analysis is simplified by the use of complex variables. A resulting expression for the square of the velocity is in agreement with Imai [same Proc. 20, 636–645 (1938)]. Some of the streamlines are plotted for $M=0$ and for $M=0.4$. *W. R. Sears.*

Imai, Isao. On the moment acting on a cylinder of arbitrary cross section placed in a uniform flow of a compressible fluid. Proc. Phys.-Math. Soc. Japan (3) 23, 939–957 (1941). [MF 15013]

Using a new procedure, the author [Rep. Aeronaut. Res. Inst. Tokyo Imp. Univ. no. 216 (1941)] studied the subsonic flow of a compressible fluid past cylinders derived from a circle by Joukowski's transformation. His results for the moment acting on the cylinder agreed with Kaplan's [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 671 (1939)] in the case of the elliptic cylinder but not in the case of the symmetrical Joukowski airfoil. In the present paper the author applies his new method to the calculation of the moment on an arbitrary cylinder placed in a uniform flow, for comparison with Kaplan's corresponding formula. The procedure, like the Poggi method employed by Kaplan, is a variant of the Janzen-Rayleigh theory in which the velocity potential is developed in ascending powers of M^2, where M is the Mach number of the undisturbed flow. Terms in M^2 are calculated. The moment is evaluated by means of Blasius' contour integral carried over a curve enclosing the cylinder. The potential is expressed as $\Phi=\Phi_0+M^2\Phi_1+\cdots$ so that Φ_0 is the potential for incompressible flow about the cylinder; Φ_1 is calculated from Φ_0 by means of an integral formula carried over from the earlier paper. By virtue of the contour integral the moment is calculated in terms of the residues of the integrand. This theory is applied to the calculation of the moment for arbitrary profiles in terms of the coefficients of the transformation that transforms them into a circle.

Finally, Kaplan's analysis is inspected and the discrepancy mentioned above is attributed to an error in evaluating a certain contour integral. In a note added at the end of the paper, the author presents formulas for the asymptotic behavior of Φ and the stream function Ψ for the compressible flow past an arbitrary cylinder, from which the moment on the cylinder can be calculated without any approximation. The derivation of these formulas is the subject of the paper reviewed below. *W. R. Sears* (Inglewood, Calif.).

Imai, Isao. On the asymptotic behaviour of compressible fluid flow at a great distance from a cylindrical body, with special reference to its lift and moment. Proc. Phys.-Math. Soc. Japan (3) 24, 231–253 (1942). [MF 15026]

In order to study the asymptotic behavior of the velocity potential Φ, the stream function Ψ and the density ρ in the case of uniform subsonic flow past a cylinder, the author assumes these quantities to be developed in powers of M^2, where M is the Mach number of the main flow, and applies a successive-approximation procedure developed in an earlier paper [see the following review]. In this procedure, the successive terms in the series are calculated by means of an integral involving the preceding ones. By induction it is proved, on the basis of this theory, that the correct asymptotic forms for Φ and Ψ are

$$\Phi=\frac{U}{2}\left\{z+\bar{z}+\frac{i\Gamma}{2\pi U}\log(z/\bar{z})+\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}A_{mn}^{l}z^{m}\bar{z}^{n}(\log z\bar{z})^{l}\right\},$$

$$\Psi=\frac{U}{2i}\left\{z-\bar{z}+i(1-M^2)\frac{\Gamma}{2\pi U}\log z\bar{z}+\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}B_{mn}^{l}z^{m}\bar{z}^{n}(\log z\bar{z})^{l}\right\},$$

where $z=x+iy$, $\bar{z}=x-iy$, x and y are coordinates parallel and perpendicular to the undisturbed flow, Γ is the circulation, U is the velocity of the undisturbed flow and A_{mn}^{l}, B_{mn}^{l} are constants. By introduction of polar coordinates $z=re^{i\theta}$, it is shown that the asymptotic form assumed by Glauert [Proc. Roy. Soc. London. Ser. A. 118, 113–119 (1928)] was not general enough, and that assumed by Bateman [Proc. Nat. Acad. Sci. U. S. A. 24, 246–251 (1938)]

was more general than necessary to represent the true conditions. The actual expressions for the coefficients of r^0, r^{-1}, and $r^{-1}\log r$ in Φ and Ψ and of r^{-1}, r^{-2}, and $r^{-2}\log r$ in ρ are next determined. Finally, the lift L, drag D and moment M_s on the cylinder are calculated by integration. It is shown that $D=0$, $L=\rho_\infty U\Gamma$ (Glauert's result), where ρ_∞ is the density of the undisturbed flow, and

$$M_s=-\pi\rho_\infty U^2(1+(1-M^2)^{\frac{1}{2}})\Im(A^0_{-1,0})$$
$$=-\pi\rho_\infty U^2(1+(1-M^2)^{-\frac{1}{2}})\Im(B^0_{-1,0}).$$

Thus, to calculate M_s it is only necessary to know the coefficient of z^{-1} in the asymptotic expression for either Φ or Ψ. The formula for M_s is shown to agree with the one deduced in the paper reviewed above, which is correct to the order M^2. *W. R. Sears* (Inglewood, Calif.).

Imai, Isao. A new method of successive approximations for dealing with the two-dimensional subsonic flow of a compressible fluid. Proc. Phys.-Math. Soc. Japan (3) **24**, 120–129 (1942). [MF 15023]

This paper is an extension of the author's previous work with Aihara [Rep. Aeronaut. Res. Inst. Tôkyô Imp. Univ. no. **194** (1940)] on the M^2 expansion method of successive approximations in the calculation of two-dimensional subsonic flows. Again he introduces complex variables, and in the relations

$$\partial\bar{F}/\partial z=(1-\rho/\rho_0)\partial\Phi/\partial z,\quad \partial\bar{F}/\partial z=i(\rho_0/\rho-1)\partial\Psi/\partial z,$$

where $F=\Phi+i\Psi$, and Φ is the velocity potential, Ψ the stream function, ρ the density, expands both sides in powers of M^2, equating terms of equal power. This reduces the calculation of all the successive approximations for F to single quadratures, and thus yields Φ and Ψ simultaneously. The arbitrary function in each quadrature can be determined from the boundary condition for the stream function only, thereby simplifying the analysis. The method is applied to the flow past a circular cylinder with circulation around it, for which the first approximation is obtained. The convergence of the method is not discussed.

D. Gilbarg (Washington, D. C.).

Imai, Isao. Note on the velocity distribution round an elliptic cylinder at high speeds. Proc. Phys.-Math. Soc. Japan (3) **26**, 71–77 (1944). [MF 15084]

The method of successive approximations of the paper reviewed above is used to derive, to order M^2, closed formulae for the velocity distribution around an arbitrary profile in terms of its conformal mapping on the unit circle. The approximation to the velocity potential is expressed as a Fourier series in the argument on the unit circle; the coefficients are determined by the theory of residues from the Cauchy integral representation of the series. A particular application is then made to the flow about an elliptic cylinder, and formulae are obtained which are essentially the same as those of Tomotika and Tamada [Rep. Aeronaut. Res. Inst. Tôkyô Imp. Univ. no. **201** (1940)] using Poggi's method of successive approximations. *D. Gilbarg.*

Hasimoto, Zirô. On the subsonic flow of a compressible fluid past an elliptic cylinder. I. Proc. Phys.-Math. Soc. Japan (3) **24**, 696–709 (1942). [MF 15038]

Hasimoto, Zirô. On the subsonic flow of a compressible fluid past an elliptic cylinder. II. Proc. Phys.-Math. Soc. Japan (3) **24**, 710–714 (1942). [MF 15039]

The method of Imai in the paper reviewed above is applied to determine the first approximation to the stream function and velocity potential. In part I the cylinder has circulation around it, and the analysis is carried out in terms of its conformal mapping on a circle. In part II the flow is without circulation and elliptic coordinates are used. The greater part of both papers is devoted to the details of fitting the boundary conditions. Agreement is obtained with the work of Imai and Aihara and of Tomotika and Tamada quoted in the preceding reviews. *D. Gilbarg.*

Hasimoto, Zirô. On the subsonic flow of a compressible fluid past a circular cylinder between two parallel walls. Proc. Phys.-Math. Soc. Japan (3) **25**, 563–574 (1943). [MF 15068]

The same problem has been treated by E. Lamla [Luftfahrtforschung **17**, 329–331 (1940); these Rev. **2**, 266]. However, the present investigation is believed to be more accurate, as the boundary conditions are satisfied exactly. The profile is not exactly circular but the deviation from circular shape is very small and is only 1% for $d/D=0.322$ where d is the diameter of the circular cylinder, D is the width of the channel. The method used is that of Janzen-Rayleigh, using a method of solving the nonhomogeneous equation for the first approximation (terms in M^2, where M is the Mach number) proposed by I. Imai [see the third preceding review]. If u_p is the maximum velocity on the profile at the top of the circle, u_b is the maximum velocity on the channel wall and U is the velocity of flow far from the cylinder, then

$$\frac{u_p}{U}=2+\frac{2}{3}\left(\frac{\pi d}{2D}\right)^2+\frac{8}{9}\left(\frac{\pi d}{2D}\right)^4+\cdots$$
$$+M^2\left\{\frac{7}{6}+\frac{43}{18}\left(\frac{\pi d}{2D}\right)^2+\frac{5981}{10080}\left(\frac{\pi d}{2D}\right)^4+\cdots\right\},$$

$$\frac{u_b}{U}=1+\left(\frac{\pi d}{2D}\right)^2+\frac{1}{3}\left(\frac{\pi d}{2D}\right)^4+\cdots$$
$$+M^2\left\{\frac{11}{6}\left(\frac{\pi d}{2D}\right)^2+\frac{79}{36}\left(\frac{\pi d}{2D}\right)^4+\cdots\right\}.$$

This result is somewhat different from that of Lamla.

H. S. Tsien (Pasadena, Calif.).

Tomotika, Susumu, and Tamada, Kô. Application of the hodograph method to the flow of a compressible fluid past an elliptic cylinder. Proc. Phys.-Math. Soc. Japan (3) **23**, 958–972 (1941). [MF 15014]

The authors have applied the Kármán-Tsien method, which uses the tangent to the adiabatic pressure-volume curve as an approximation to the curve, to the compressible nonviscous flow over an inclined elliptic cylinder. The distortion of the profile during the transformation from incompressible flow to compressible flow is found to consist of a thickening of the profile together with a decrease in the angle of attack. The critical Mach number for the first appearance of sonic flow velocity and the aerodynamic moment are then calculated without considering the effect of these profile distortions. The critical Mach number is lower than and the moment is greater than that calculated by the first order solution with the Janzen-Rayleigh method made by the authors previously [Rep. Aeronaut. Res. Inst. Tôkyô Imp. Univ., no. **201** (1940)]. The moment is, in fact, higher than that predicted by the Glauert-Prandtl approximation. This is used as the reason for the authors' statement that the Kármán-Tsien method is not suitable for calculating the moment of a body in compressible flow.

H. S. Tsien (Pasadena, Calif.).

Suzuki, Masaru. On the force and moment acting on an infinite cylinder moving in a flow of an incompressible perfect fluid with uniform vorticity. Proc. Phys.-Math. Soc. Japan (3) **24**, 313–330 (1942). [MF 15028]

Following Tasiro [same Proc. **20**, 91–99 (1938)], the author adopts a coordinate system moving with the cylinder and determines an expression for the pressure in terms of the stream function and the velocities and accelerations of the cylinder, assuming uniform vorticity. Restricting what follows to cases where the stream function of the undisturbed flow is an integral expression of the second degree in the fixed coordinates, he is able to construct a formula for the stream function of the total flow. The force and moment on the cylinder are then calculated. The formula for the moment does not agree with Ray's [Proc. Roy. Soc. London Ser. A. **158**, 522–535 (1937); Z. Angew. Math. Mech. **18**, 223–232 (1938)]; the author attributes the discrepancy to errors in the earlier paper.

The general results are used to set up the equations of motion for an elliptic cylinder moving freely in a flow of uniform vorticity; these are integrated for (i) a flat plate, (ii) an elliptic cylinder in a uniform shear flow and (iii) an elliptic cylinder moving in a vorticial flow of a fluid of the same density as the cylinder. Next, the force and moment on a flat-plate airfoil placed in a uniform shear flow are determined. [The reviewer finds that these results agree with Tsien's [Quart. Appl. Math. **1**, 130–148 (1943); these Rev. **5**, 21]; note that Tsien's equation (60) for the moment coefficient contains a typographical error.] The corresponding centers of pressure are calculated and numerical results are shown graphically.

Finally, the force and moment on a flat-plate airfoil, placed in a fluid whose undisturbed motion is a solid-body rotation, are calculated; these are the same as for an airfoil describing a circular path at constant angle of attack in a fluid otherwise at rest, and are well known.

W. R. Sears (Inglewood, Calif.).

Suzuki, Masaru. On the motion of a doubly symmetric cylinder with a constant circulation round it through an incompressible perfect fluid. Proc. Phys.-Math. Soc. Japan (3) **25**, 116–138 (1943). [MF 15052]

The two-dimensional motion under no external force is discussed, and the equations of motion are integrated completely in terms of elliptic and elementary functions. According to the values of integration constants and the constants of the section of the cylinder, three cases are distinguished, two of which result in periodic motions. Many paths are calculated numerically and are shown by figures.

D. Gilbarg (Washington, D. C.).

Yosida, Yosikatu. On the problem of potential flow about the bodies of revolution. Proc. Phys.-Math. Soc. Japan (3) **25**, 413–424 (1943). [MF 15062]

The author is concerned with the flow of an otherwise undisturbed stream of a perfect incompressible fluid past a symmetrically oriented surface of revolution. The problem is solved by determining rigorously that continuous distribution of sources along the axis of symmetry which together with the uniform flow will yield the given surface as a streamline. A polar coordinate system is adopted and the surface is defined by giving the distance from origin to surface as a power series in $\cos\theta$, the cosine of the polar angle. A Fredholm integral equation of the first kind with a nonsymmetric kernel is obtained and this kernel is expanded in Legendre polynomials of argument $\cos\theta$. The conventional trigonometric Fourier expansion of the resulting equation then leads to an infinite set of linear algebraic equations for the X_i, the coefficients of the source distribution function. These equations have the form $\sum_{i=1}^{\infty} K_{ij}X_i = b_j$, where each K_{ij} is itself an infinite sum. No specific cases are discussed.

G. F. Carrier (Cambridge, Mass.).

Omara, M. A. Hydrodynamic forces on a moving cylinder in presence of vortices. Proc. Math. Phys. Soc. Egypt **1**, no. 3, 15–18 (1939). [MF 14105]

In an earlier paper [Philos. Mag. (7) **27**, 200–211 (1939)] the author has considered the two-dimensional accelerated motion of a cylinder in an incompressible perfect fluid at rest, obtaining formulae for the force and moment which are generalizations of the well-known Blasius integral formulae. In this paper he proceeds to the case in which there may be isolated vortices in the fluid. Carrying over results of the earlier paper, he repeats the calculations for force and moment, assuming that the cylinder has no cusps on its contour and that the total circulation (circulation about the cylinder plus sum of vortex strengths) does not vary with time. Finally, he considers briefly the effects of cusps on the profile; as in the earlier paper, these are shown to introduce additional forces.

W. R. Sears (Inglewood, Calif.).

Loitzansky, L. G. Approximate calculation of the turbulent boundary layer on an aerofoil profile. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 433–448 (1945). (Russian. English summary) [MF 15438]

The parameters taken as characteristic of the turbulent boundary layer are a generalization of the Prandtl-Bury parameters for the case of any Reynolds number. With the help of the theorem of impulses, the author arrives at a general differential equation of the problem. This equation becomes closed if certain analogies between turbulent and laminar layers are assumed. The problem is reduced to a simple quadrature. The influence of the laminar range can easily be taken into account. As an illustration the author gives numerical calculations and a comparison with the results obtained through experiment.

Author's summary.

Feinsilber, A. M. On some fundamental relations in aerodynamics of the turbulent boundary layer in compressible and incompressible fluids. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 541–542 (1945). [MF 14419]

The author presents an expression for the shear distribution in the turbulent boundary layer for compressible or incompressible flow:

$$\tau/\tau_0 = 1 - \int_0^t (du_1/dt)dt \int_0^t \rho_1 u_1 dt \Big/ \int_0^1 (du_1/dt)dt \int_0^t \rho_1 u_1 dt,$$

where $u_1 = u/\bar{U}$, $\rho_1 = \rho/\bar{\rho}$, $t = y/\delta$. He outlines the application of the formula to flow past a flat plate for incompressible motion.

H. W. Liepmann (Pasadena, Calif.).

Rocard, Yves. Sur les relations existantes entre la couche limite et les tourbillons de Bénard-Kármán. Rev. Sci. (Rev. Rose Illus.) **78**, 371–373 (1940). [MF 13317]

The author attempts to determine the two independent parameters in von Kármán's formula for drag coefficient in the case of vortex streets behind a circular cylinder. By considering the generation of vorticity and making certain assumptions, he obtains the value 1.47 for the drag coefficient of a circular cylinder, as compared with the experimental value 1.2–1.5.

C. C. Lin (Providence, R. I.).

Nikolsky, A. A. Plane parallel flow with free boundaries past a vortex. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 425–440 (1944). (Russian. English summary) [MF 12583]

The author investigates the passage of a jet of incompressible fluid around a vortex. Denoting by $w(z)$ the complex potential, he considers the image of the flow pattern in the ζ-plane, where $\zeta=1/(dw/dz)$. The function $\zeta(z)$ maps the domain in which the flow is defined into the exterior of a circular arc, the point k where the vortex is located being transformed into $\zeta=0$. Combining several simple transformations, the author maps the above domain into the unit circle $|Z|<1$ so that k is mapped into the origin, and the two points at infinity of the jet into two points of the boundary of the circle. The function

$$\lambda(Z)=w(z(Z))+(\Gamma/2\pi i)\log Z$$

maps the circle $|Z|<1$ into an infinite strip, $c_1<\Im(\lambda)<c_2$, c_1, c_2 being constants. Using this result the author determines $w(z)$. It depends on parameters which correspond to the difference of the velocity directions at $+\infty$ and $-\infty$ and the location of the vortex. The shape of the jet in the physical plane, the distribution of velocity, and the forces acting on the vortex are determined. *S. Bergman.*

Lin, C. C. On the stability of two-dimensional parallel flows. III. Stability in a viscous fluid. Quart. Appl. Math. 3, 277–301 (1946). [MF 14522]

This is the last part of the author's theory of the stability of plane laminar motion. [For parts I and II, cf. the same Quart. 3, 117–142, 218–234 (1945); these Rev. 7, 225, 226.] The stability character of a viscous fluid is considered in detail. The author proceeds first to give a proof of a criterion of stability due to Heisenberg: if a velocity profile has an "inviscid" neutral disturbance with nonvanishing wave number and phase velocity, the disturbance with the same wave number is unstable in the real fluid when the Reynolds number is sufficiently large. This destabilizing effect of viscosity is one of the most interesting phenomena in the general stability theory; its physical and mathematical significance is carefully discussed.

The author then discusses the behavior of the so-called neutral curve $c_i(\alpha, R)=0$ for the two characteristic types of velocity distribution, the boundary layer type profile and the symmetrical profile. The asymptotic behavior of the neutral curve is discussed first. The main difference between profiles with and without a point of inflection is that the two branches of the neutral curve approach $\alpha=\alpha_s$ and $\alpha=0$ for profiles with a flex, but both converge to $\alpha=0$ for the profile without a flex. The most important results are as follows. (i) All symmetrical and boundary layer profiles are unstable for sufficiently large Reynolds number R. (ii) There always exists a minimum R below which the motion is stable. A similar result was obtained by Synge from energy considerations. Synge found a limiting (α, R) curve below which the motion is necessarily stable. The author's discussion of the asymptotic behavior of the $c_i(\alpha, R)$ curves shows further that there always exists a maximum value of α beyond which the motion is stable for all Reynolds numbers. Hence the qualitative shape of the $c_i(\alpha, R)=0$ curve is determined.

The author proceeds to show that simple approximate expressions for the stability limit can be obtained from his general analysis for a given velocity profile. These approximate stability limits for plane Poiseuille flow and Blasius flow are found to be $R=5906$ and $R=502$. The Reynolds numbers are based on the width of the channel and the displacement thickness, respectively. Finally, the method for computing the complete instability curve is presented and the plane Poiseuille case and the Blasius problem worked out in detail. The stability limit for Blasius flow had been given before by Tollmien and Schlichting. The present more exact computations agree well with Tollmien's result as far as the minimum critical Reynolds number is concerned. The value found here is $R=420$. The neutral curve for Poiseuille motion had not been obtained before. The minimum critical number here is found to be $R=5314$. The agreement with the estimate from the simple criterion mentioned above is thus very good.

A discussion of the physical significance of the viscous effects and of future developments concludes the paper.

H. W. Liepmann (Pasadena, Calif.).

Rouse, Hunter. A general stability index for flow near plane boundaries. J. Aeronaut. Sci. 12, 429–431 (1945). [MF 13700]

The author attempts to develop a general stability index on the assumption that the only parameters which decide the stability of laminar flow are the normal distance y from the solid boundary, the velocity gradient dv/dy and the kinematic viscosity coefficient. He then arrives at the stability index $\chi=y^2(dv/dy)/\nu$ and concludes that a laminar flow is stable or unstable according to whether the maximum value of χ is less than or above a critical value, which is of the order of a few hundred. [Reviewer's remark. The author does not explain whether his critical limit refers to the "upper" critical Reynolds number above which some infinitesimal disturbances are self-excited or the "lower" critical Reynolds number below which turbulent motion is impossible. These two may be as much as a hundred times different from each other. For another discussion of this paper, see H. W. Liepmann, same J. 13, 94 (1946).]

C. C. Lin (Providence, R. I.).

Chou, P. Y. Pressure flow of a turbulent fluid between two infinite parallel planes. Quart. Appl. Math. 3, 198–209 (1945). [MF 13532]

The author applies his general equations of turbulence [Quart. Appl. Math. 3, 38–54 (1945); these Rev. 6, 246] to the case of turbulent flow through a channel. The calculated mean velocity distribution and average square of fluctuations of turbulent velocity agree well with experimental data available. *C. C. Lin* (Providence, R. I.).

Hu, N. Velocity and temperature distributions in turbulent wakes behind an infinite cylinder and a body of revolution. Chinese J. Phys. 5, 1–29 (1944). [MF 13758]

Chou's theory of turbulence [cf. the preceding review] is applied to investigate the velocity and temperature distributions in turbulent wakes behind an infinite cylinder and a body of revolution. The results of the theory for the mean squares of velocity fluctuations across the wake and for the distributions of mean velocity and temperature behind the wakes are found to agree well with the experiments of Fage, of Fage and Falkner, and of Hall and Hislop.

Author's summary.

Hu, N. Velocity and temperature distributions in turbulent wakes behind a row of equally spaced parallel rods and behind a square grid. Chinese J. Phys. 5, 30–48 (1944). [MF 13759]

Chou's theory of turbulence [cf. the preceding reviews for references] is applied successfully to the wakes behind a

system of equal distant parallel rods and a square grid. It is found that both the velocity and temperature distributions in the first kind of wake are the same cosine functions of the coordinates across the wake and those for the second kind of wake are just the superpositions of the wakes behind two perpendicular systems of parallel rods. In both cases Taylor's law of linear decay of turbulent fluctuation and quadratic increase of scale of microturbulence are obtained.
Author's summary.

Huang, Su-Shu. The turbulent jet. Chinese J. Phys. **5**, 105–123 (1944). [MF 13763]

The theory of the spread of turbulent jets, based upon Chou's improved general theory of turbulence [Quart. Appl. Math. **3**, 38–54 (1945); these Rev. **6**, 246] is given in the present paper. Starting from the differential equations for mean motion and the equations of double correlations for both velocity and temperature fluctuations, the author has calculated the mean velocity and the mean temperature distributions in the two-dimensional, in the axially symmetrical and in the half jet. In all these cases the same procedure of approximation has been followed. The results are compared with experimental data, and the agreement between theory and experiment is quite good.
Author's summary.

Chang, S. L. The turbulent flow through a circular pipe. Chinese J. Phys. **5**, 124–137 (1944). [MF 13764]

Chou's general theory of turbulence [Quart. Appl. Math. **3**, 38–54 (1945); these Rev. **6**, 246] is applied to the turbulent flow through a circular pipe. The method of solution employed by Chou in his investigation of the turbulent flow through a channel is used here and leads to results which agree well with experiment. *Author's summary.*

Nevzgljadov, V. G. A contribution to the statistical theory of turbulence. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 466–468 (1945). [MF 14423]

The author considers Reynolds' equations for turbulent flow. The variables in these equations are divided into two groups: "fundamental" variables and "reducible" variables. The relations between the reducible variables, that is, essentially the fluctuating quantities, and the fundamental variables are called "equations of state." The fundamental problem of turbulence is, of course, to establish these "equations of state." The author approaches the problem from the point of view of statistical mechanics. It is shown that with certain limitations the "equations of state" are obtainable from a condition similar to the entropy consideration in statistical mechanics. *H. W. Liepmann.*

Nevzgliadov, V. G. A contribution to the phenomenological theory of turbulence. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 165–168 (1945). [MF 13737]

The author considers a phenomenological theory of turbulence by regarding the average velocity, the average pressure, and the average kinetic energy of turbulent fluctuations as the fundamental variables. The other average quantities appearing in the averaged equations of motion, equation of continuity, and equation of energy are assumed to be dependent on these fundamental quantities. To establish these "dynamical state equations," the concept of "mixing length" is again introduced. The author further assumes the mixing length to be proportional to the velocity of fluctuation, thus resulting in exchange coefficients proportional to the average kinetic energy of turbulence. He then compared his equations with those given by Kolmogoroff [Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **6**, 56–58 (1942)], who did not make the last assumption but included the equation for mean vorticity, not considered by the present author. *C. C. Lin.*

Nevzgljadov, V. A phenomenological theory of turbulence. Acad. Sci. USSR. J. Phys. **9**, 235–243 (1945). [MF 15435]

Burgers, J. M. Considerations on the statistical theory of turbulent flow. Nederl. Tijdschr. Natuurkunde **8**, 5–18 (1941). (Dutch) [MF 15364]

Cârstoiu, I. Instabilité d'un tourbillon dans un canal plan. Bull. École Polytech. Bucarest [Bul. Politechn. Bucureşti] **16**, 19–24 (1945). [MF 15623]

Roy, Maurice. Sur l'écoulement adiabatique permanent d'un gaz parfait autour d'un dièdre indéfini. C. R. Acad. Sci. Paris **218**, 866–869 (1944). [MF 15332]

Roy, Maurice. Sur la structure de l'onde de choc, limite d'une quasi-onde de choc dans un fluide compressible et visqueux. C. R. Acad. Sci. Paris **218**, 813–816 (1944). [MF 15327]

Matsumoto, Toshizô. Ein anderer Beweis des Hilfssatzes im Problem über Grundwasserströmung. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **24**, 89–91 (1944). [MF 15554]

Carleman, T. Sur la propagation d'un mouvement à la surface libre d'un liquide. Ark. Mat. Astr. Fys. **32B**, no. 8, 2 pp. (1945). [MF 15097]

The author proves that, if a blast of wind disturbs a part of the free surface of a body of water which is initially at rest and considered to be incompressible, the disturbance is propagated instantaneously over the entire free surface.
H. S. Tsien (Pasadena, Calif.).

Dean, W. R. On some cases of the reflexion of surface waves by an inclined plane barrier. Proc. Cambridge Philos. Soc. **42**, 24–28 (1946). [MF 14408]

The paper considers the reflection of a simple harmonic train of surface waves by a rigid plane barrier inclined at an angle $\alpha=\pi/2n$, n an integer, to the undisturbed free surface. The mathematical problem is the determination of a harmonic function $u(x, y)$ with (a) vanishing normal derivative for a ray making the angle α with the x-axis and (b) $\partial u/\partial y - Cu = C'$ for $y=0$. Write $F(z)=u+iv$. Conditions (a) and (b) become conditions on F and dF/dz and a general expression is found for $F(z)$, for α defined as above. The coefficients in the expression for the free surface are computed for $n=3$. [The mathematical formulation is the usual one of a first approximation. This approximation implies that the ratio of the displacement of the free surface to the depth is very small, which is not true near the barrier.]
D. G. Bourgin (Urbana, Ill.).

Craig, Richard A. A solution of the nonlinear vorticity equation for atmospheric motion. J. Meteorol. **2**, 175–178 (1945). [MF 15599]

Assuming that friction and vertical velocity can be neglected, that the motion takes place in a horizontal plane and that the atmospheric density does not vary in this plane, the velocity can be expressed by a stream function ψ. If

the x-axis points towards E, the y-axis towards north and if $\beta=(2\omega\cos\phi)/R$, where ω is the angular velocity of the earth's rotation, ϕ the latitude, R the earth's radius, the stream function is given by the equation

$$(1)\qquad \left(\frac{\partial}{\partial t}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi+\beta\frac{\partial\psi}{\partial x}=0.$$

The function ψ will satisfy this equation if $\nabla^2\psi=f(\psi)$ and $\partial f/\partial t+\beta\partial f/\partial x=0$. Of meteorological interest is a periodic solution of the form

$$(2)\qquad \psi=C\exp\{2\pi i(x-ct)/L+2\pi iy/D\},$$

where C is an arbitrary constant, L the wave length, c the wave velocity and D the lateral extent. In this case the following relation must exist between the parameters according to (1): $c=-(\beta L^2/4\pi^2)(1+L^2/D^2)$. This expression does not depend on the speed of the general zonal current, contrary to the results obtained by Rossby [J. Marine Res. 2, 38–55 (1939)] and Haurwitz [J. Marine Res. 3, 35–50 (1940)], who linearized (1) by assuming a small perturbation superimposed on a zonal current. Because of the nonlinear character of (1), superposition of solutions of the form (2) is possible only if

$$(2\pi/L_1)^2+(2\pi/D_1)^2=(2\pi/L_2)^2+(2\pi/D_2)^2=\cdots.$$

It is also possible to solve (1) by an expression of the form

$$\psi=C_1f(y)+C_2\exp\{2\pi i(x-ct)/L+2\pi iy/D\}$$

provided that $f'[(2\pi/L)^2+(2\pi/D)^2]+f'''=0$, which implies that a connection must exist between the strength of the north-south distribution and the dimensions of the waves in the westerlies.

Similar computations are carried out for the case of a spherical earth. *B. Haurwitz* (Cambridge, Mass.).

Sakadi, Zyuro. Reinvestigation on the theory of sea and land breeze. Proc. Phys.-Math. Soc. Japan (3) **25**, 624–632 (1943). [MF 15075]

In contrast to some other investigations the effects of the earth's rotation, the compressibility of the atmosphere and the work done by the expansion of the atmosphere are taken into account. After developing the general equations of the problem in a rotating coordinate system, the deviations from the equilibrium state caused by the sea breeze circulation are assumed small enough to permit the linearization of the differential equations. While the coefficients of the resulting linear system of equations are variable, their variation is so small that they may be regarded as constants. The solutions can then be assumed to be a combination of exponential and trigonometric functions, provided that certain relations are satisfied between the parameters appearing in these solutions. These relations are given explicitly, but no meteorological interpretation of the result is attempted. *B. Haurwitz* (Cambridge, Mass.).

Cahn, Albert, Jr. An investigation of the free oscillations of a simple current system. J. Meteorol. **2**, 113–119 (1945). [MF 14553]

The author investigates a problem of inertial oscillations of ocean currents, initiated by C. G. Rossby [J. Marine Res. 1, 239–263 (1938)]. It concerns the motion in the open sea due to a parallel current of finite width suddenly generated by impulsive forces. Let x and y be coordinates in the plane of the ocean bed, with x in the direction of the current and u, v the corresponding velocity components. If t is the time and if the constant depth D_0 of the ocean is large compared with the wave elevation h, then

$$\partial u/\partial t=fv,\quad \partial v/\partial t=-fu-g\partial h/\partial y,\quad \partial h/\partial t+D_0\partial v/\partial y=0,$$

where $f=2\Omega\sin\varphi$ is the Coriolis parameter, Ω is the angular velocity of the earth and φ is the latitude. This set of equations is solved under the initial conditions that at $t=0$, $v=0$, $h=0$, $u=u_0(y)$. The appropriate solution is that of the "telegraph equation" and can be written as

$$v=-1/(2\lambda)\int_{-f\Omega}^{f\Omega}u_0(y-\alpha)J_0\{(f^2t^2-\alpha^2/\lambda^2)^{\frac{1}{2}}\}d\alpha,$$

where $\lambda=(gD_0f)^{\frac{1}{2}}$. If the initial current is of width $2a$ with constant velocity U, then $u_0=U$ for $|y|<a$, $u_0=0$ for $|y|>a$. The oscillation of the current is then traced in detail. The amplitude of the waves is found to decrease as t^{-1} because of the spreading of the waves. The period of oscillation is found to be always less than 12 pendulum hours but approaches this value as $t\to\infty$. This is in agreement with the investigations of Kelvin [see H. Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, pp. 317 ff.] and Rossby. *H. S. Tsien* (Pasadena, Calif.).

Kusakov, M. Capillary gravitational waves at the interface between two viscous liquids of finite depth. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **14**, 232–239 (1944). (Russian) [MF 12224]

The author considers the problem of capillary gravitational waves of infinitely small amplitude on the boundary line of two viscous fluids of finite depth. He neglects the inertia terms in the Navier-Stokes equations so that the equations for the velocity vector become linear. Applying the usual methods, that is, writing the potential and stream function in the form $\sum A_\nu\exp\{k_\nu z+ik_\nu x+\omega t\}$, he determines by extensive but simple computations the constants A_ν, K_ν, ω. The damping coefficient, the amplitude of the waves and the speed of their propagation are determined. *S. Bergman* (Providence, R. I.).

Miche. Mouvements ondulatoires de la mer en profondeur constante ou décroissante. Forme limite de la houle lors de son déferlement. Application aux digues maritimes. Ann. Ponts Chaussées **1944** (114e année), 25–78, 131–164, 270–292, 369–406 (1944). [MF 14692]

This is a comprehensive treatment of two-dimensional surface wave motion. The fluid is assumed to be nonviscous and incompressible. For this type of problem, the usual method of investigation adopts the Eulerian equations. However, due to the facts that the boundary condition at the free surface is nonlinear and the solution of the Eulerian equations does not give the path of the fluid particles directly, this method does not offer any advantages over the Lagrangian equations, especially if the method of successive approximation is used. If x, y are the coordinates at time t of the fluid particle which at $t=0$ is located at x_0, y_0, p is the pressure, ρ the density, h a measure for the amplitude and g is the gravitational constant, then

$$(1)\qquad \begin{aligned} x&=x_0+h\varphi_1+h^2\varphi_2+h^3\varphi_3+\cdots,\\ y&=y_0+h\psi_1+h^2\psi_2+h^3\psi_3+\cdots,\\ p/\rho&=gy_0+h\chi_1+h^2\chi_2+h^3\chi_3+\cdots. \end{aligned}$$

By substituting (1) into the Lagrangian equations and then equating terms of equal powers in h, differential equations for the functions $\varphi(x_0, y_0, t)$, $\psi(x_0, y_0, t)$ and $\chi(x_0, y_0, t)$ are determined. The equations for φ_1, ψ_1 and χ_1 are homogeneous and constitute the first approximation to the problem. The equations for the second approximation φ_2, ψ_2, and χ_2 are

nonhomogeneous with the particular integrals involving φ_2, ψ_2 and χ_2. If $G(x_0, y_0, t)$ is harmonic in x_0 and y_0, the first approximation is given by

(2) $\varphi_1=\partial G/\partial x_0$, $\psi_1=\partial G/\partial y_0$, $\chi_1=g\psi_1-\partial^2 G/\partial t^2+f_1(t)$;

G and f can be determined by boundary conditions.

For progressive waves in a channel of uniform depth H, half wave length L and period T,

$$(3)\qquad G=\frac{\cosh a(H-y_0)}{a\sinh aH}\cos(bt-ax_0),$$

where $a=\pi/L$, $b=\pi/T$. For this case the value of $f_1(t)$ in (2) is zero. To this order of approximation, the fluid motion is irrotational. Only in the second approximation is the effect of vorticity felt. In the second approximation two types of waves are studied in detail, the Stokes irrotational wave and Boussinesq's waves of zero residual horizontal motion of fluid particles. By superimposing two progressive waves of type (3) travelling in opposite directions, the first order solution for standing waves is obtained. The complete solution for second order terms shows that the previous work of Sainflou [same Ann. **98**, 5–48 (1928)] is not correct. The difference is, however, not large.

For $H\to\infty$, the G-function for standing waves is

$$2a^{-1}\sin bt\, e^{-ay_0}\sin ax_0.$$

The solution for standing waves in a sloping sea with "shore line" at the origin and slope α with respect to the x-axis can be obtained by superimposing the above G-function with its various reflections with respect to the x-axis and to the line of sea bottom, under the restriction that $\alpha=\pi/2m$, m an integer. If the polar coordinates (s, β) are used for (x_0, y_0), the first order solution for standing waves is

$$(4)\qquad G=2a^{-1}\sin bt\cdot g(x_0, y_0),$$

$$g=(\pi/2\alpha)^{\frac{1}{2}}\sum_0^\infty\frac{(-as)^n}{n!}\frac{A_n}{\sin n\alpha}\cos n(\alpha-\beta),$$

where $A_{n+1}=A_n\cot n\alpha$, $A_1=1$ and $(A_n/\sin n\alpha)_{n=0}=A_1/\cos\alpha$. This solution is evidently regular at the origin. For small values of α, there is the following approximate form for g:

$$(5)\qquad g'=(\pi/2\alpha)^{\frac{1}{2}}J_0(2\sigma),\quad \sigma=(\pi s/\alpha L)^{\frac{1}{2}}.$$

By using the asymptotic form of J_0, g can be written approximately as

$$(6)\qquad g''=\cos(2\sigma-\pi/4)/(2\alpha\sigma)^{\frac{1}{2}}.$$

The author divides the motion into four regions. For local depth $H'>L$, the waves are regularly spaced and behave like the waves of a sea of very great depth. For $L/\pi<H'<L$, the energy per wave length decreases from the value E_l for great depth. The wave length and the wave amplitude also decrease up to 9%. For $\pi\alpha^2L/4<H'<L/\pi$ the solution g'' applies. The wave length decreases and the energy per wave length diminishes to almost $\frac{1}{2}E_l$. The wave amplitude, however, increases as the shore is approached. For $H'<\pi\alpha^2L/4$, g' is useful for calculation; here the "wave" is very much distorted and the amplitude is the largest of all the regions.

Corresponding to g' and g'', there is the solution h' and h'' obtained by replacing the Bessel function of the first kind by the Bessel function of the second kind. This solution has a logarithmic singularity at the origin, the "shore." By superimposing g and h, the author obtains the first order solution for a progressive wave in shallow water of constant slope $\alpha=\pi/2m$, m an integer. A study of the energy flux across any section of the "sea" gives the conclusion that waves cannot be propagated towards the shore unless a source of energy such as wind is present. Then the flux of energy toward the shore is finally destroyed by the breaking of surf. The analysis also leads to a method of treating approximately waves in water of varying depth by using the solution of constant depth, adjusted to the local depth in the problem.

The paper concludes with a section on the limiting form of waves, rotational or irrotational, in a sea of constant finite depth. The solution is then applied to the breaking of surf in a sloping sea. *H. S. Tsien* (Pasadena, Calif.).

Burgers, J. M. On the influence of the concentration of a suspension upon the sedimentation velocity (in particular for a suspension of spherical particles). Nederl. Akad. Wetensch., Proc. **44**, 1045–1051, 1177–1184 (1941); **45**, 9–16, 126–128 (1942). [MF 15229]

The mutual interaction of small particles settling in a viscous fluid has essentially two aspects. (1) The vertical pressure gradient in the fluid is increased by the weight of the suspended particles. The author proposes without further proof to take care of this effect by introducing in the formulas for the settling velocity the specific gravity of the suspension instead of that of the liquid. However, he does not elaborate on the case of sediment mixtures where this procedure is not necessarily correct. (2) The main part of the paper is devoted to the effect of the suspended particles on the viscosity of the fluid as it affects the settling of a given particle. The author finds for this correction the same expression as has been given previously by Einstein except for a slightly smaller constant which may be explained as follows. The viscosity is increased in a suspension because all shearing motion is prevented wherever fluid is replaced by solid particles. The probability is smaller for a grain to cover a point in the immediate neighborhood of the grain under consideration than any other point of its flow field because the different grains cannot penetrate one another. As the most intense shearing motion in the field of a settling particle is always near its surface, the viscosity of a suspension is not increased as effectively in the field of a settling particle as in the general case described by the Einstein formula. *H. A. Einstein* (Pasadena, Calif.).

Scholte, J. G. On the relation between ocean depressions and microseisms. I, II. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 669–683 (1943). (Dutch. German, English and French summaries) [MF 15785]

Rudnick, I. Acoustic transmission through a fluid lamina. J. Acoust. Soc. Amer. **17**, 245–253 (1946). [MF 14671]

The acoustic wave equation

$$\nabla\cdot(c^2\nabla\log\rho)=D^2\log\rho/Dt^2+\gamma(\nabla\times\mathbf{V})^2,$$

velocity of sound $c^2=\gamma p/\rho$, $D/Dt=\partial/\partial t+\mathbf{V}\cdot\nabla$, is derived for a fluid medium under adiabatic conditions. For uniform motion and relative excess pressure $p'/p_0=(p-p_0)/p_0\ll 1$, it reduces to $(c^2\nabla^2-D^2/Dt^2)p'=0$. Plane waves in a medium at rest (density ρ) impinging on the plane interface $x=0$ with a medium moving with uniform velocity V_y and density ρ_1 are considered. Snell's law is unchanged if the plane of incidence is perpendicular to z, but becomes $c_1/\sin\vartheta_1-V_y=c/\sin\vartheta$ if it is perpendicular to y, indicating a critical angle of incidence given by $\sin\vartheta=c/(c_1-V_y)$ beyond which an "inhomogeneous wave" and thus total reflection result. If the moving medium consists of a lamina of thickness l, partial reflection and transmission take place at both undercritical and overcritical incidences. The final expressions

for transmission and reflection coefficients as functions of the parameters ϑ, $l\omega/c_1$, V_g/c and ρ_1/ρ, where ω denotes circular frequency, are derived by the standard method. They are specialized for the case of a gas where the difference in lamina density is solely due to that of temperature. This case is investigated experimentally. Experimental results agree satisfactorily with the idealized theory (the temperature distribution across the lamina being bell-shaped rather than rectangular). *H. G. Baerwald.*

Korringa, J., Kronig, R., and Smit, A. On the theory of the reflection of sound by porous media. Physica 11, 209–230 (1945).

A simple model for the porous medium is used. This consists of a large number of rigid identical spheres in contact and arranged in a simple cubic lattice. Propagation of a sound wave moving along one of the cubic axes is considered. Internal friction and the effect of heat conduction are taken into account. The spheres are assumed to have an infinite heat conductivity and the wave length of the sound in free air is considered to be large compared with the lattice constant and the radius of the spheres. There are four equations of motion: the equation of motion taking into account the viscosity of the air; the equation of continuity; the equation of state of the gas; the thermo-elastic equation relating the transformation of mechanical energy into heat. These equations are all linearized; that is, displacements, velocity, etc. are assumed to deviate only slightly from the equilibrium values.

The boundary conditions are that the velocity **u** must vanish on the surface of the spheres and that the temperature at the surface of the spheres must be the equilibrium temperature. Solution of these equations is accomplished in the usual fashion by splitting the velocity into irrotational and divergenceless terms so that the problem is referred to scalar and vector potentials rather than the velocity. The method used for the solution is due to Rayleigh [Philos. Mag. (5) **34**, 481–502 (1892) = Scientific Papers, vol. 4, Cambridge University Press, 1903, pp. 19–38] and consists in breaking up the potential in the neighborhood of one sphere into two terms: a scattered term consisting of a diverging wave and a term consisting of standing waves, presumably being due to the contribution of the scattering of all the other spheres at the sphere being considered. By making use of the periodicity relations of a lattice, it is then possible to solve the problem.

Comparison is made with experimental results in which the acoustical impedance for normal incidence of a sound wave on a porous acoustical material of finite thickness on a rigid back was measured. The agreement with experiment was satisfactory. *H. Feshbach* (Cambridge, Mass.).

Elasticity, Plasticity

Herman, B. Some theorems of the theory of anisotropic media. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 89–92 (1945). [MF 15223]

By introducing complex coordinates, the author gives a simple proof of the following theorem. If a medium has a rotating axis of symmetry of order N (that is, if rotation through $2\pi/N$ is a covering operation), then the medium is axially isotropic for all physical properties defined by tensors of order (or rank) $0, 1, \cdots, N-1$. Applications to heat conduction, electromagnetism, piezoelectricity, magnetostriction, elasticity and photoelasticity are indicated. The case where the medium has two axes of symmetry is also discussed. *J. L. Synge* (Columbus, Ohio).

Prager, W. On plane elastic strain in doubly-connected domains. Quart. Appl. Math. **3**, 377–380 (1946). [MF 14528]

The problem of plane strain with given boundary stresses, formulated in terms of Airy's stress function, leads to a solution containing a number of constants which, for multiply connected domains, must be determined and which can be determined from the condition that the displacements must be single-valued. The author shows that the equations determining these constants are also the natural boundary conditions of the variational problem for the stress function. This remark is of importance when the direct methods of the calculus of variations are used to determine the stress function for a doubly-connected domain. *E. Reissner.*

Weinel, E. Die Spannungserhöhung durch Kreisbogenkerben. Z. Angew. Math. Mech. **21**, 228–230 (1941). [MF 15854]

Der ebene Spannungszustand in einem von zwei Kreisbogen begrenzten Bereich wird mit Hilfe von Bipolarkoordinaten durch Fourierintegrale dargestellt. Als Anwendung wird die Spannungserhöhung in Kreisbogenkerben und -Löchern durchgerechnet. *Author's summary.*

Rivaud, Jacques. Remarques sur le problème de l'élasticité non linéaire. C. R. Acad. Sci. Paris **218**, 698–700 (1944). [MF 15323]

Bergeot, Pierre. Extension du théorème de Clapeyron aux déformations finies. C. R. Acad. Sci. Paris **218**, 903–904 (1944). [MF 15239]

Sobrero, L. A proposito dell'interpretazione meccanica delle funzioni di variabile ipercomplessa. Boll. Un. Mat. Ital. (2) **5**, 90–95 (1943). [MF 16092]

Gasparini, Ida. Sopra una proprietà caratteristica dei sistemi isotropi. Boll. Un. Mat. Ital. (2) **5**, 13–18 (1943). [MF 16080]

Si dà una dimostrazione diretta del fatto che nella teoria ordinaria dell'elasticità la coincidenza di ogni triedro principale di deformazione con un triedro principale di tensione basta da sola a caratterizzare i sistemi isotropi. *Author's summary.*

de Beauregard, Olivier Costa. Sur les équations fondamentales, classiques, puis relativistes, de la dynamique des milieux continus. J. Math. Pures Appl. (9) **23**, 211–217 (1944). [MF 15740]

Versluys, W. A. Damped plane vibrations of a homogeneous string under the influence of an external force. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 69–73 (1943). (Dutch. German, English and French summaries) [MF 15774]

Haag, Jules. Sur la théorie des fils élastiques. C. R. Acad. Sci. Paris **218**, 865–866 (1944). [MF 15331]

Announcement of extensions of the author's work in Ann. Sci. École Norm. Sup. (3) **46**, 105–129 (1929). *E. Reissner* (Cambridge, Mass.).

Rakhmatulin, H. A. Inclined impact on a flexible cord. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 449–462 (1945). (Russian. English summary) [MF 15439]

The author deals with the problem of the propagation of waves along an infinite flexible cord, set up by the inclined impact of a material point, which preserves its velocity after the shock. The problem is solved for both elastic and plastic deformations, friction between the cord and the striking solid being taken into account.

From the author's summary.

Favre, Henry. Les vibrations transversales des cordes pesantes verticales. Schweiz. Bauztg. **122**, 253–254, 285–287 (1943). [MF 15540]

The free vibrations of a heavy vertically suspended string are discussed. The linear variation in tension along the string, which is due to this orientation, is taken into account, and a linear wave equation with variable coefficients is obtained. The first approximations to the possible motions are taken in the form $y=(1-\epsilon x)f(t-(1-\epsilon x)x/c)$, corresponding to the classical constant tension solution $y=f(t-x/c)$. The solutions are interpreted both for pulse propagation and standing wave phenomena. The results are valid only for very small amplitudes and small values of a dimensionless parameter which is essentially the ratio of string density to average tension. *G. F. Carrier.*

Fadle, J. Bemerkung zu: Eine Anwendung des absoluten Parallelismus auf die Schalentheorie, von F. Reutter. Z. Angew. Math. Mech. **24**, 48 (1944). [MF 15850]

The paper appeared in the same Z. **22**, 87–98 (1942); these Rev. **4**, 230.

Levy, Samuel. Large-deflection theory of curved sheet. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **895**, 24 pp. (5 plates) (1943). [MF 14395]

The equations of the finite deflection theory of curved cylindrical sheets are solved for the practically interesting case of a sheet buckled by loads in the direction of the axis of the cylinder. The solution is taken in the form of trigonometric double series. For the coefficients in these series a system of simultaneous nonlinear equations is obtained. The assumption that all the coefficients except the two most significant ones may be taken equal to zero permits a relatively simple solution for the specific problem considered by the author. Some calculations carried out with six coefficients retained indicate that the two-term solution is satisfactory in the range of parameter values covered in the paper. Numerical results are given in the form of curves for the ratio of deflection to sheet thickness and for the "effective width" of the sheet as a function of the ratio of edge strain to initial buckling strain and of the degree of initial curvature of the sheet. *E. Reissner* (Cambridge, Mass.).

Goldenweiser, A. L. Qualitative investigation of stressed states in thin-walled shells. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 463–478 (1945). (Russian. English summary) [MF 15440]

The paper contains an exposition of the fundamental equations of thin shells referred to an arbitrary Gaussian coordinate system. The equations are given in tensor form and the general integrals of the complete system of equations in stresses are analyzed qualitatively when the thickness of the shell is assumed to be small. The author investigates the errors arising from the use of the Kirchhoff-Love hypothesis and outlines several methods of approximate integration of the system of equations of the theory of shells. *I. S. Sokolnikoff* (Madison, Wis.).

Feodosiev, V. I. Large displacements and stability of a circular membrane with fine corrugations. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 389–412 (1945). (Russian. English summary) [MF 15348]

A thin corrugated membrane with flat center (such as in an aneroid barometer) is subjected to external loads. The author derives the equilibrium equations for axially symmetric shells with large deflections, specializes them to membranes with fine sinusoidal corrugations and solves them by Galerkin's method. The first approximation indicates the influence on stability of the flat center and of the joints used in connecting the corrugated portion of the membrane to the flat center. The second approximation exhibits the local loss in stability in the external corrugation. This paper simplifies and generalizes earlier results by D. Y. Panov [same journal **5**, 303–318 (1941); these Rev. **4**, 231] and considers some of the simpler examples of stability of corrugated membranes. The paper contains extensive computations. *I. S. Sokolnikoff* (Madison, Wis.).

Favre, Henry. Contribution à l'étude des plaques obliques. Schweiz. Bauztg. **120**, 35–36, 51–54, 60 (1942). [MF 15539]

To facilitate the treatment of problems of transverse bending of thin elastic plates having the shape of a parallelogram, the author transforms the basic equations of the theory to oblique coordinates u, v related to Cartesian coordinates x, y by $x=u+v\cos\alpha$, $y=v\sin\alpha$. As an example of the application of his formulas he gives an approximate solution by Ritz's method for the problem of the plate which is built-in along all four sides and carries a uniformly distributed load. *E. Reissner* (Cambridge, Mass.).

Kurdumov, A. A. Solution by polynomials of the plane problem of the theory of elasticity for rectangular anisotropic plates. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 339–342 (1945). (Russian. English summary) [MF 15344]

The author seeks a solution of the differential equation for the stress function $\varphi(x, y)$ for an anisotropic strip bounded by $y=\pm b$, in the form $\varphi(x, y)=\sum_{k=0}^{n} f_k(y)x^k$, where the functions $f_k(y)$ satisfy a system of ordinary differential equations of the fourth order. The prescribed distribution of stress along the boundaries $y=\pm b$ is assumed to be given by polynomials in x. If n is properly chosen, the resultant force and the resultant moment acting in the section $x=l$ can be made equal to the preassigned values. The method of solution is illustrated by considering a strip of length l which carries a distributed uniform load on the edge $y=b$ and which is bent by a concentrated load directed along the y-axis and applied at the end $x=l$. *I. S. Sokolnikoff.*

Sherman, D. I. Stresses in an anisotropic elastic homogeneous medium formed by flush fitting of bodies. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 347–352 (1945). (Russian. English summary) [MF 15346]

Let S_0 be a two-dimensional multiply connected domain bounded by an exterior contour L and m interior contours γ_k. The contours γ_k bound simply connected domains S_k ($k=1, \cdots, m$), which correspond to the holes in a plane anisotropic plate occupying the domain S_0. If the holes are filled by forcing into them m discs whose elastic properties

are identical with those of the plate, what is the resulting distribution of stresses and deformations in the domain $S_0+\sum_{k=1}^{n}S_k$? It is supposed that the external forces acting on the contour L are specified and the discontinuities in displacements and stresses across the contours γ_k are known. The author reduces the solution of the problem to the determination of two sets of analytic functions of the complex variables $z_1=x+\mu_1 y$ and $z_2=x+\mu_2 y$, where μ_1 and μ_2 are complex constants depending on the characteristics of the medium. These two sets of functions are determined with the aid of the integrals of Cauchy's type.

I. S. Sokolnikoff (Madison, Wis.).

Isibasi, Tadasi. Stresses in a semi-infinite plate with a circular notch under uniform tension. Mem. Fac. Engineering, Kyushu Imp. Univ. **9**, 131–143 (1940). [MF 15552]

Tranter, C. J., and Craggs, J. W. The stress distribution in a long circular cylinder when a discontinuous pressure is applied to the curved surface. Philos. Mag. (7) **36**, 241–250 (1 plate) (1945). [MF 15747]

Higgins, Thomas James. Stress analysis of shafting exemplified by Saint-Venant's torsion problem. Experimental Stress Analysis **3**, 94–101 (1945).

The author gives a useful survey of known solutions of the torsion problem for bodies of revolution. The bibliography contains 46 items. *W. Prager* (Providence, R. I.).

Sokolovsky, W. W. Plastic torsion of a shaft of circular cross section and variable diameter. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 343–346 (1945). (Russian. English summary) [MF 15345]

The author points out that the problem of plastic equilibrium in a body of revolution under torsion is statically determinate. A stress function is introduced which takes care of the yield condition. The equation of equilibrium then leads to a partial differential equation of parabolic type; its characteristics are the orthogonal trajectories of the lines of resulting shear stress. A stepped shaft and a conical shaft are treated as examples. As in all so-called statically determinate problems of equilibrium, the boundary of the plastic region remains indeterminate.

W. Prager (Providence, R. I.).

Zwolinsky, N. V. Rayleigh's waves in an inhomogeneous elastic half-space of special type. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **9**, 261–278 (1945). (Russian. English summary) [MF 15353]

The author considers the free oscillating motion of an inhomogeneous half-space where Lamé's coefficients and density are the same exponential functions of depth. On the boundary of the half-space no stresses exist. At infinite depth displacements and stresses are zero. The velocities of seismic rays in the medium are constant. The author succeeds in carrying through the investigations for a medium where Poisson's coefficient is zero. It is found that the given medium has a normal dispersion. It is shown, besides, that Love's waves do not exist in this medium.

From the author's summary.

Matthews, T. H. An elementary proof of a theorem in geophysics. Trans. Roy. Soc. Canada. Sect. III. (3) **39**, 21–23 (1945). [MF 15519]

In a flat earth in which the seismic velocity increases linearly with depth, the quickest wave paths between points on the surface are arcs of circles, as are also the wave fronts from a point source. The simple proof presented illustrates principles in several branches of mathematics.

A. Blake (Aberdeen Proving Ground, Md.).

Glagolev, N. I. Resistance of cylindrical bodies in rolling. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 318–333 (1945). (Russian. English summary) [MF 15342]

The problem of rolling elastic cylindrical bodies over an elastic foundation is considered. The line of contact of two rolling discs is assumed to consist of one range of adhesion and two ranges of sliding. The author sets up a basic integral equation and solves it by the method of Carleman [Ark. Mat. Astr. Fys. **16**, no. 26 (1922)]. The impossibility of the existence of a range of adhesion and but one range of sliding along the line of contact is demonstrated. The last section of the paper is devoted to the solution of the case of a single range of sliding along the entire line of contact.

H. P. Thielman (Ames, Iowa).

Sokolovsky, W. W. The theory of plasticity—outline of work done. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 495–508 (1945). (Russian. English summary) [MF 15444]

Sokolovsky, W. W. The theory of plasticity—outline of work done in Russia. J. Appl. Mech. **13**, A-1–A-10 (1946). [MF 15736]

[The second paper is a translation of the first.] Summary of an address at the Science Congress of the Allied Nations in Moscow and Leningrad, June, 1945. The paper lists various types of problems which were attacked by Russian scientists in recent years, and gives a survey of the more important results. The extensive bibliography will prove very useful. *W. Prager* (Providence, R. I.).

BIBLIOGRAPHICAL NOTES

★Coker, W. C., Editor. Studies in Science. The University of North Carolina Press, Chapel Hill, N. C., 1946. viii+375 pp. $3.00.

This is part of the Sesquicentennial Publications of the University of North Carolina. It also appeared as volume 61 of J. Elisha Mitchell Sci. Soc. The mathematical papers contained in this volume were reviewed in these Rev. 7, 47, 75, 81, 101, 108.

Summa Brasiliensis Mathematicæ.

This journal, published in Rio de Janeiro by the Fundação Getúlio Vargas, is to appear in separate numbers at irregular intervals. Vol. 1, no. 1 appeared in December, 1945.